Trigonometric Functions 1

Time : to : Date Name

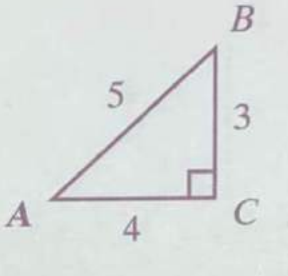
100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Given a right triangle, $\triangle ABC$, with C as the right angle,

$$\sin A = \frac{3}{5}$$

$$\cos A = \frac{4}{5}$$

$$\tan A = \frac{3}{4}$$



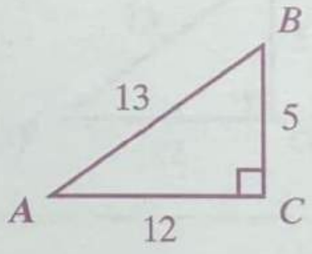
Terminology: $\sin A$, $\cos A$ and $\tan A$ are respectively the *sine*, *cosine* and *tangent* functions of angle A.

Complete and check your answers.

$$\sin A = \frac{5}{13}$$

$$\cos A = \frac{12}{13}$$

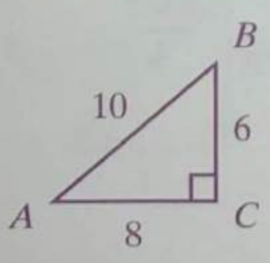
$$\tan A = \frac{5}{12}$$



Answers: $\frac{5}{13}, \frac{12}{13}, \frac{5}{12}$

Find the sin, cos and tan of angle A in each of the following exercises.

(1)

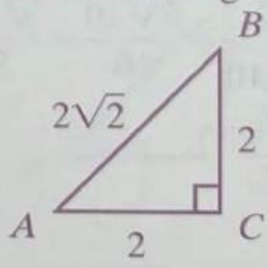


$$\sin A = \frac{6}{\boxed{10}} = \frac{3}{5}$$

$$\cos A = \frac{8}{10} = \frac{4}{5}$$

$$\tan A = \frac{6}{8} = \frac{3}{4}$$

(2)



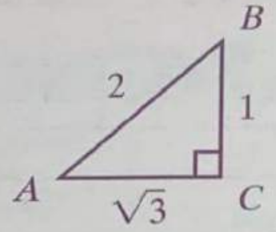
$$\sin A = \frac{2}{\boxed{2\sqrt{2}}} = \frac{\sqrt{2}}{2}$$

$$\cos A = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan A = \frac{2}{2} = 1$$

M 1 b

(3)

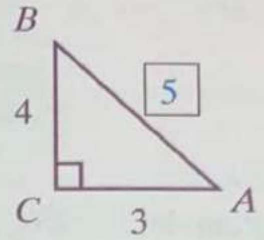


$\sin A = \frac{1}{2}$

$\cos A = \frac{\sqrt{3}}{2}$

$\tan A = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

(5)

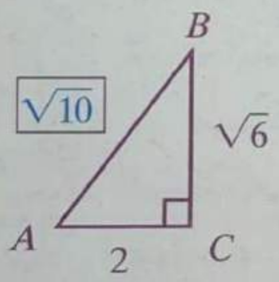


$\sin A = \frac{4}{5}$

$\cos A = \frac{3}{5}$

$\tan A = \frac{4}{3}$

(4)

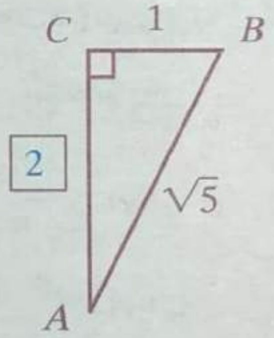


$\sin A = \frac{\sqrt{6}}{\sqrt{10}} = \frac{2\sqrt{15}}{10} = \frac{\sqrt{15}}{5}$

$\cos A = \frac{2}{\sqrt{10}} = \frac{2\sqrt{10}}{10} = \frac{\sqrt{10}}{5}$

$\tan A = \frac{\sqrt{6}}{2}$

(6)



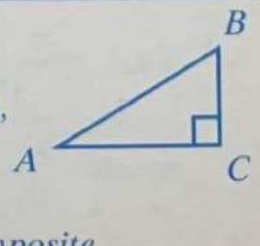
$\sin A = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$

$\cos A = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$

$\tan A = \frac{1}{2}$

Terminology:

Given a right triangle, $\triangle ABC$ with C as the right angle,
AC is the side *adjacent* to angle A, BC is the side *opposite* of angle A,
AB is the *hypotenuse*.



$$\sin A = \frac{opposite}{hypotenuse}, \quad \cos A = \frac{adjacent}{hypotenuse}, \quad \tan A = \frac{opposite}{adjacent}$$

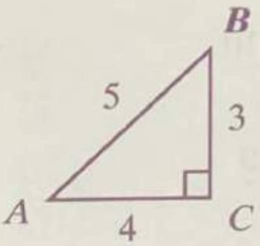
Trigonometric Functions 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

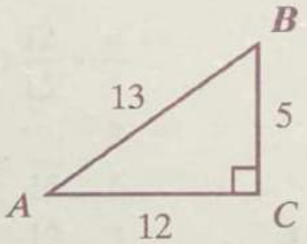
Given a right triangle, $\triangle ABC$, with C as the right angle,

$$\sin B = \frac{4}{5}$$
$$\cos B = \frac{3}{5}$$
$$\tan B = \frac{4}{3}$$



Complete and check your answers.

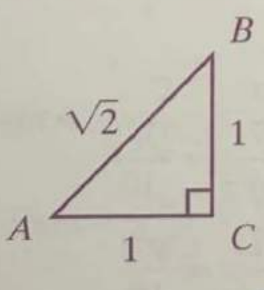
$$\sin B = \frac{12}{13}$$
$$\cos B = \frac{5}{13}$$
$$\tan B = \frac{12}{5}$$



Answers: $\frac{12}{13}$, $\frac{5}{13}$, $\frac{12}{5}$

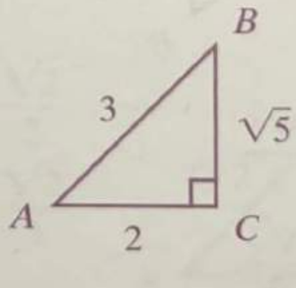
Find the sin, cos and tan of angle B in each of the following exercises.

(1)



$$\sin B = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\cos B = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\tan B = \frac{1}{1} = 1$$

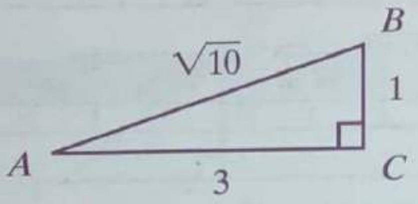
(2)



$$\sin B = \frac{2}{3}$$
$$\cos B = \frac{\sqrt{5}}{3}$$
$$\tan B = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

M 2 b

(3)

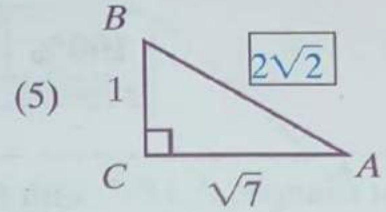


$$\sin B = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$$

$$\cos B = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$$

$$\tan B = \frac{3}{1} = 3$$

(5)

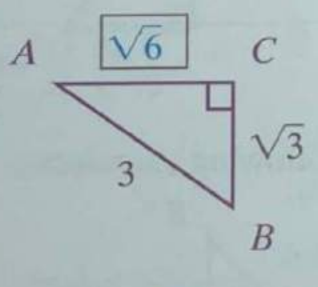


$$\sin B = \frac{\sqrt{7}}{2\sqrt{2}} = \frac{\sqrt{14}}{4}$$

$$\cos B = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

$$\tan B = \frac{\sqrt{7}}{1} = \sqrt{7}$$

(4)

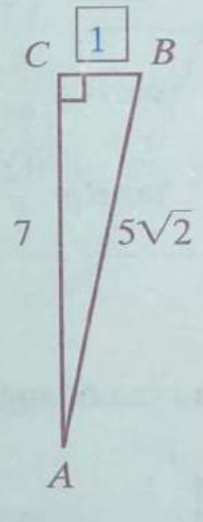


$$\sin B = \frac{\sqrt{6}}{3}$$

$$\cos B = \frac{\sqrt{3}}{3}$$

$$\tan B = \frac{\sqrt{6}}{\sqrt{3}} = \frac{\sqrt{18}}{3} = \frac{3\sqrt{2}}{3} = \sqrt{2}$$

(6)



$$\sin B = \frac{7}{5\sqrt{2}} = \frac{7\sqrt{2}}{10}$$

$$\cos B = \frac{1}{5\sqrt{2}} = \frac{\sqrt{2}}{10}$$

$$\tan B = \frac{7}{1} = 7$$

M 3 a Trigonometric Functions 1

Time : to : Date Name

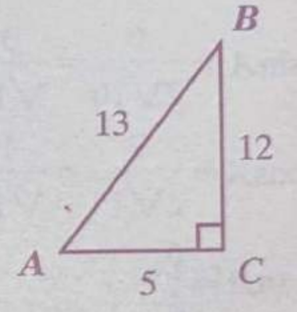
100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

Find the sin, cos and tan of angles **A** and **B** as shown in the example.

Ex. $\sin A = \frac{12}{13}$ $\sin B = \frac{5}{13}$

$\cos A = \frac{5}{13}$ $\cos B = \frac{12}{13}$

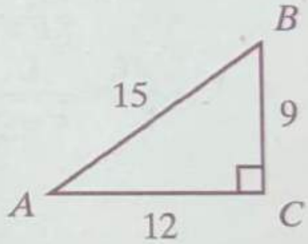
$\tan A = \frac{12}{5}$ $\tan B = \frac{5}{12}$



(1) $\sin A = \frac{9}{15} = \frac{3}{5}$ $\sin B = \frac{12}{15} = \frac{4}{5}$

$\cos A = \frac{12}{15} = \frac{4}{5}$ $\cos B = \frac{9}{15} = \frac{3}{5}$

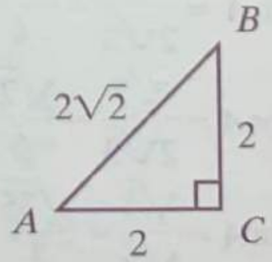
$\tan A = \frac{9}{12} = \frac{3}{4}$ $\tan B = \frac{12}{9} = \frac{4}{3}$



(2) $\sin A = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\sin B = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}$

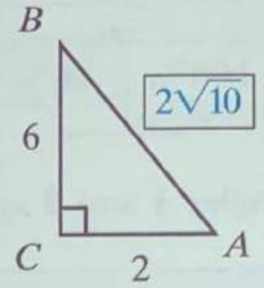
$\cos A = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\cos B = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\tan A = \frac{2}{2} = 1$ $\tan B = \frac{2}{2} = 1$



M 3 b

(3)

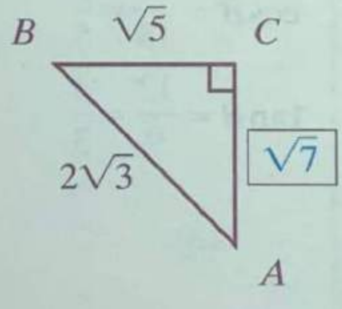


$$\sin A = \frac{6}{2\sqrt{10}} = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10} \qquad \sin B = \frac{2}{2\sqrt{10}} = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$$

$$\cos A = \frac{2}{2\sqrt{10}} = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10} \qquad \cos B = \frac{6}{2\sqrt{10}} = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$$

$$\tan A = \frac{6}{2} = 3 \qquad \tan B = \frac{2}{6} = \frac{1}{3}$$

(4)



$$\sin A = \frac{\sqrt{5}}{2\sqrt{3}} = \frac{\sqrt{15}}{6} \qquad \sin B = \frac{\sqrt{7}}{2\sqrt{3}} = \frac{\sqrt{21}}{6}$$

$$\cos A = \frac{\sqrt{7}}{2\sqrt{3}} = \frac{\sqrt{21}}{6} \qquad \cos B = \frac{\sqrt{5}}{2\sqrt{3}} = \frac{\sqrt{15}}{6}$$

$$\tan A = \frac{\sqrt{5}}{\sqrt{7}} = \frac{\sqrt{35}}{7} \qquad \tan B = \frac{\sqrt{7}}{\sqrt{5}} = \frac{\sqrt{35}}{5}$$

M 4 a

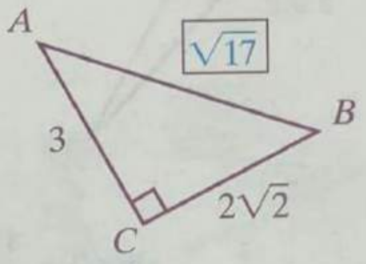
Trigonometric Functions 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

Find the sin, cos and tan of angles ***A*** and ***B***.

(1)



$\sin A = \dfrac{2\sqrt{2}}{\sqrt{17}} = \dfrac{2\sqrt{34}}{17}$

$\cos A = \dfrac{3}{\sqrt{17}} = \dfrac{3\sqrt{17}}{17}$

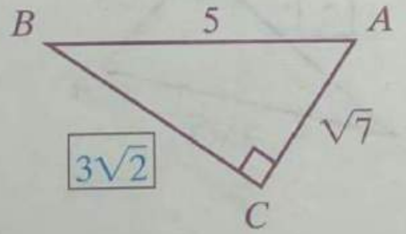
$\tan A = \dfrac{2\sqrt{2}}{3}$

$\sin B = \dfrac{3}{\sqrt{17}} = \dfrac{3\sqrt{17}}{17}$

$\cos B = \dfrac{2\sqrt{2}}{\sqrt{17}} = \dfrac{2\sqrt{34}}{17}$

$\tan B = \dfrac{3}{2\sqrt{2}} = \dfrac{3\sqrt{2}}{4}$

(2)



$\sin A = \dfrac{3\sqrt{2}}{5}$

$\cos A = \dfrac{\sqrt{7}}{5}$

$\tan A = \dfrac{3\sqrt{2}}{\sqrt{7}} = \dfrac{3\sqrt{14}}{7}$

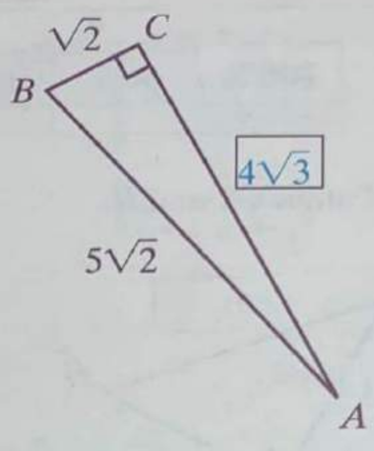
$\sin B = \dfrac{\sqrt{7}}{5}$

$\cos B = \dfrac{3\sqrt{2}}{5}$

$\tan B = \dfrac{\sqrt{7}}{3\sqrt{2}} = \dfrac{\sqrt{14}}{6}$

M 4 b

(3)



$$\sin A = \frac{\sqrt{2}}{5\sqrt{2}} = \frac{1}{5}$$

$$\cos A = \frac{4\sqrt{3}}{5\sqrt{2}} = \frac{4\sqrt{6}}{10} = \frac{2\sqrt{6}}{5}$$

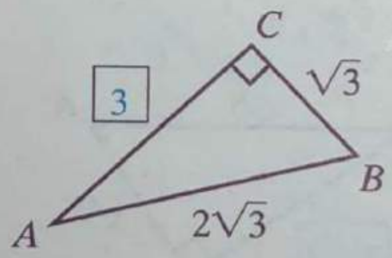
$$\tan A = \frac{\sqrt{2}}{4\sqrt{3}} = \frac{\sqrt{6}}{12}$$

$$\sin B = \frac{4\sqrt{3}}{5\sqrt{2}} = \frac{4\sqrt{6}}{10} = \frac{2\sqrt{6}}{5}$$

$$\cos B = \frac{\sqrt{2}}{5\sqrt{2}} = \frac{1}{5}$$

$$\tan B = \frac{4\sqrt{3}}{\sqrt{2}} = \frac{4\sqrt{6}}{2} = 2\sqrt{6}$$

(4)



$$\sin A = \frac{\sqrt{3}}{2\sqrt{3}} = \frac{1}{2}$$

$$\cos A = \frac{3}{2\sqrt{3}} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}$$

$$\tan A = \frac{\sqrt{3}}{3}$$

$$\sin B = \frac{3}{2\sqrt{3}} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}$$

$$\cos B = \frac{\sqrt{3}}{2\sqrt{3}} = \frac{1}{2}$$

$$\tan B = \frac{3}{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$$

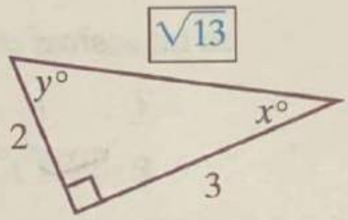
Trigonometric Functions 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

Find the sin, cos and tan of the indicated angles in each of the following exercises.

(1)



$\sin x° = \dfrac{2}{\sqrt{13}} = \dfrac{2\sqrt{13}}{13}$

$\cos x° = \dfrac{3}{\sqrt{13}} = \dfrac{3\sqrt{13}}{13}$

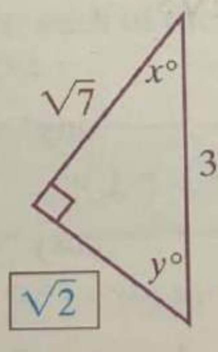
$\tan x° = \dfrac{2}{3}$

$\sin y° = \dfrac{3}{\sqrt{13}} = \dfrac{3\sqrt{13}}{13}$

$\cos y° = \dfrac{2}{\sqrt{13}} = \dfrac{2\sqrt{13}}{13}$

$\tan y° = \dfrac{3}{2}$

(2)



$\sin x° = \dfrac{\sqrt{2}}{3}$

$\cos x° = \dfrac{\sqrt{7}}{3}$

$\tan x° = \dfrac{\sqrt{2}}{\sqrt{7}} = \dfrac{\sqrt{14}}{7}$

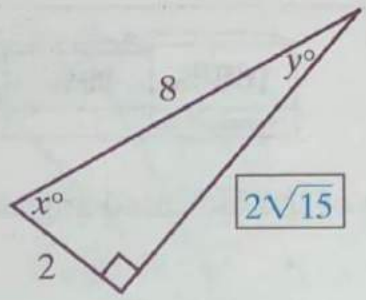
$\sin y° = \dfrac{\sqrt{7}}{3}$

$\cos y° = \dfrac{\sqrt{2}}{3}$

$\tan y° = \dfrac{\sqrt{7}}{\sqrt{2}} = \dfrac{\sqrt{14}}{2}$

M 5 b

(3)

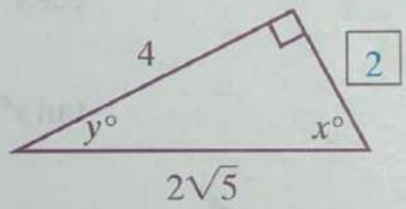


$$\sin x^\circ = \frac{2\sqrt{15}}{8} = \frac{\sqrt{15}}{4} \qquad \sin y^\circ = \frac{2}{8} = \frac{1}{4}$$

$$\cos x^\circ = \frac{2}{8} = \frac{1}{4} \qquad \cos y^\circ = \frac{2\sqrt{15}}{8} = \frac{\sqrt{15}}{4}$$

$$\tan x^\circ = \frac{2\sqrt{15}}{2} = \sqrt{15} \qquad \tan y^\circ = \frac{2}{2\sqrt{15}} = \frac{1}{\sqrt{15}} = \frac{\sqrt{15}}{15}$$

(4)



$$\sin x^\circ = \frac{4}{2\sqrt{5}} = \frac{4\sqrt{5}}{10} = \frac{2\sqrt{5}}{5} \qquad \sin y^\circ = \frac{2}{2\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$\cos x^\circ = \frac{2}{2\sqrt{5}} = \frac{\sqrt{5}}{5} \qquad \cos y^\circ = \frac{4}{2\sqrt{5}} = \frac{4\sqrt{5}}{10} = \frac{2\sqrt{5}}{5}$$

$$\tan x^\circ = \frac{4}{2} = 2 \qquad \tan y^\circ = \frac{2}{4} = \frac{1}{2}$$

Note Summary:

From exercises (1)–(4), note that $\sin x^\circ = \cos y^\circ$

$$\cos x^\circ = \sin \boxed{y^\circ}$$

$$\tan x^\circ = \frac{1}{\boxed{\tan y^\circ}}$$

Answers: y°, $\tan y^\circ$

Trigonometric Functions 1

Time : to : Date Name

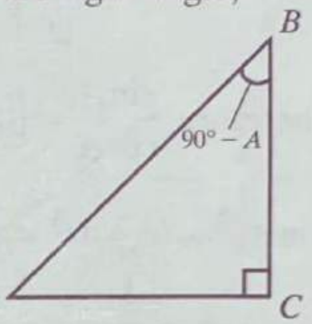
100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

From the Triangle-Sum Theorem in Level I, $m\angle A + m\angle B + m\angle C = 180°$.
Therefore, given a right triangle, $\triangle ABC$, with C as the right angle,

$$m\angle A + m\angle B = 90°$$

Given angle A, B can be expressed as:

$$B = 90° - A \ \ldots ①$$



Since $\sin A = \cos B$

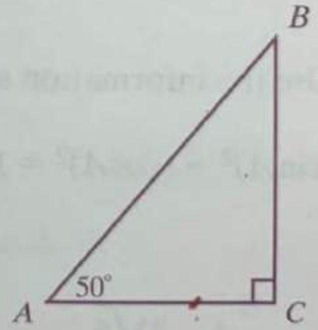
Substituting ①, $\sin A = \cos(90° - A)$

Similarly, since $\cos A = \sin B$ and $\tan A = \dfrac{1}{\tan B}$,

Substituting ①, $\cos A = \boxed{\sin(90° - A)}$ and $\tan A = \dfrac{1}{\boxed{\tan(90° - A)}}$

Answers: $\sin(90° - A)$, $\tan(90° - A)$

1. Given $A = 50°$, express each of the following functions in terms of $40°$, as shown in the example.



Ex. $\sin 50° = \cos(90° - 50°) = \cos 40°$

(1) $\cos 50° = \sin(90° - 50°) = \sin 40°$

(2) $\tan 50° = \dfrac{1}{\tan(90° - 50°)} = \dfrac{1}{\tan 40°}$

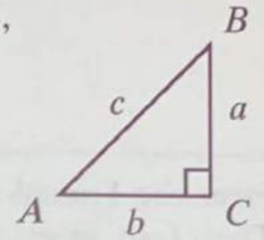
2. Given $A = 20°$, express each of the following functions in terms of $70°$.

(1) $\sin 20° = \cos(90° - 20°) = \cos 70°$

(2) $\cos 20° = \sin(90° - 20°) = \sin 70°$

(3) $\tan 20° = \dfrac{1}{\tan(90° - 20°)} = \dfrac{1}{\tan 70°}$

Given a right triangle, $\triangle ABC$, with C as the right angle,



$$\frac{\sin A}{\cos A} = \frac{\frac{a}{c}}{\boxed{\frac{b}{c}}} = \boxed{\frac{a}{b}} \quad \text{and} \quad \tan A = \boxed{\frac{a}{b}}$$

Therefore, $\dfrac{\sin A}{\cos A} = \boxed{\tan A}$ . . . ①

$$(\sin A)^2 + (\cos A)^2 = \left(\boxed{\frac{a}{c}}\right)^2 + \left(\boxed{\frac{b}{c}}\right)^2$$

$$= \frac{\boxed{a^2+b^2}}{c^2}$$

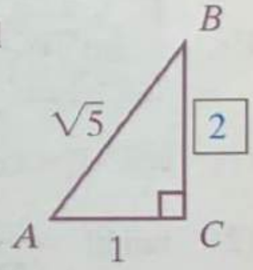
From the Pythagorean Theorem, $a^2 + b^2 = c^2$

$$= \frac{\boxed{c^2}}{c^2} = \boxed{1}$$

Therefore, $(\sin A)^2 + (\cos A)^2 = \boxed{1}$ . . . ②

Answers: $\frac{b}{c}, \frac{a}{b}, \frac{a}{b}$, $\tan A$, $\frac{a}{c}, \frac{b}{c}$, $a^2 + b^2$, c^2, 1, 1

3. Use the information above to show that $\dfrac{\sin A}{\cos A} = \tan A$, and $(\sin A)^2 + (\cos A)^2 = 1$ for the given triangle on the right.



(1) $\dfrac{\sin A}{\cos A} = \dfrac{\frac{2\sqrt{5}}{5}}{\frac{\sqrt{5}}{5}} = 2$ and $\tan A = 2$ $\quad \therefore \dfrac{\sin A}{\cos A} = \tan A$

(2) $(\sin A)^2 + (\cos A)^2 = \left(\dfrac{2\sqrt{5}}{5}\right)^2 + \left(\dfrac{\sqrt{5}}{5}\right)^2 = \dfrac{4}{5} + \dfrac{1}{5} = 1$

$\therefore (\sin A)^2 + (\cos A)^2 = 1$

Trigonometric Functions 1

Time : to : Date Name

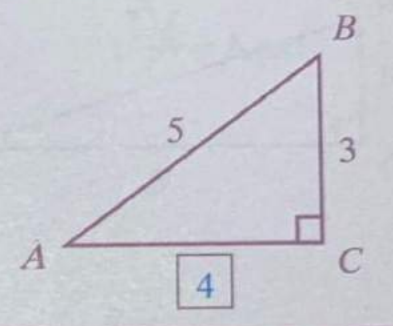
100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Ex. Given $\sin A = \frac{3}{5}$, find the values of $\cos A$ and $\tan A$.

[Sol] $AC = \sqrt{5^2 - 3^2} = \boxed{4}$

$\cos A = \frac{\boxed{4}}{5}$

$\tan A = \frac{3}{\boxed{4}}$



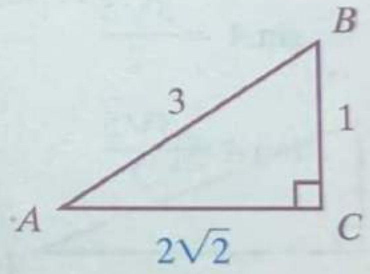
Answers: all 4

(1) Given $\sin A = \frac{1}{3}$, find the values of $\cos A$ and $\tan A$.

[Sol] $AC = \sqrt{3^2 - 1^2} = 2\sqrt{2}$

$\cos A = \frac{2\sqrt{2}}{3}$

$\tan A = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$

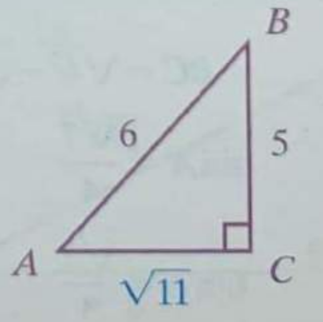


(2) Given $\sin A = \frac{5}{6}$, find the values of $\cos A$ and $\tan A$.

[Sol] $AC = \sqrt{6^2 - 5^2} = \sqrt{11}$

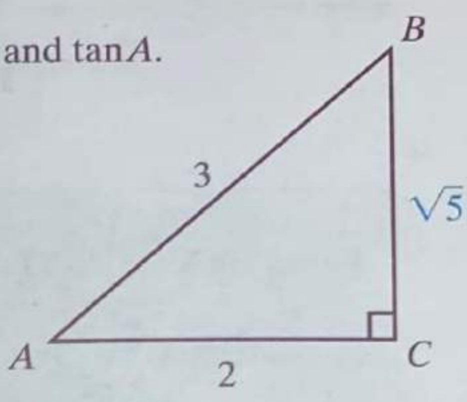
$\cos A = \frac{\sqrt{11}}{6}$

$\tan A = \frac{5}{\sqrt{11}} = \frac{5\sqrt{11}}{11}$



M 7 b

(3) Given $\cos A = \dfrac{2}{3}$, find the values of $\sin A$ and $\tan A$.

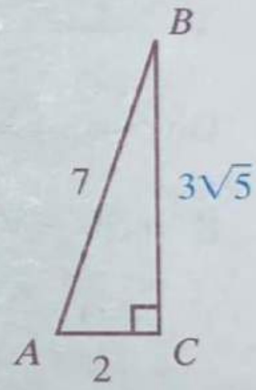


[Sol] $BC = \sqrt{3^2 - 2^2} = \sqrt{5}$

$\sin A = \dfrac{\sqrt{5}}{3}$

$\tan A = \dfrac{\sqrt{5}}{2}$

(4) Given $\cos A = \dfrac{2}{7}$, find the values of $\sin A$ and $\tan A$.

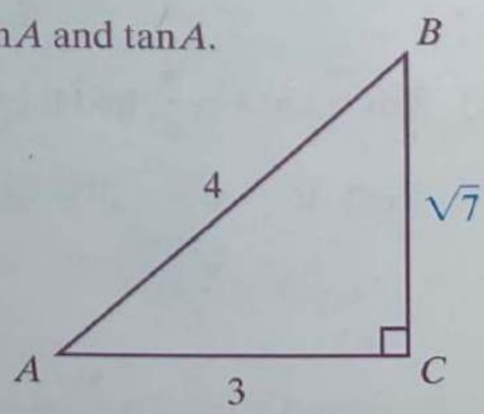


[Sol] $BC = \sqrt{7^2 - 2^2} = 3\sqrt{5}$

$\sin A = \dfrac{3\sqrt{5}}{7}$

$\tan A = \dfrac{3\sqrt{5}}{2}$

(5) Given $\cos A = 0.75$, find the values of $\sin A$ and $\tan A$.



[Sol] $\cos A = 0.75 = \dfrac{3}{4}$

$BC = \sqrt{4^2 - 3^2} = \sqrt{7}$

$\sin A = \dfrac{\sqrt{7}}{4}$

$\tan A = \dfrac{\sqrt{7}}{3}$

Trigonometric Functions 1

Time : to : Date Name

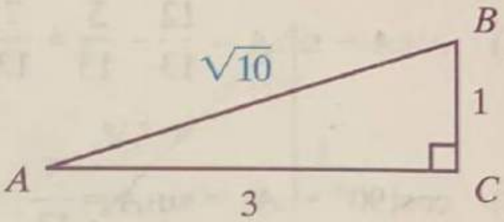
100%	90%	80%	70%	69%~
(mistakes) 0	1	2	3	4~

1. Given $\tan A = \frac{1}{3}$, find the values of $\sin A$ and $\cos A$.

[Sol] $AB = \sqrt{3^2 + 1^2} = \sqrt{10}$

$$\sin A = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$$

$$\cos A = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$$



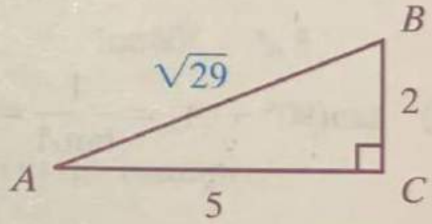
2. Given $\tan A = 0.4$, find the values of $\sin A$ and $\cos A$.

[Sol] $\tan A = 0.4 = \frac{2}{5}$

$$AB = \sqrt{5^2 + 2^2} = \sqrt{29}$$

$$\sin A = \frac{2}{\sqrt{29}} = \frac{2\sqrt{29}}{29}$$

$$\cos A = \frac{5}{\sqrt{29}} = \frac{5\sqrt{29}}{29}$$

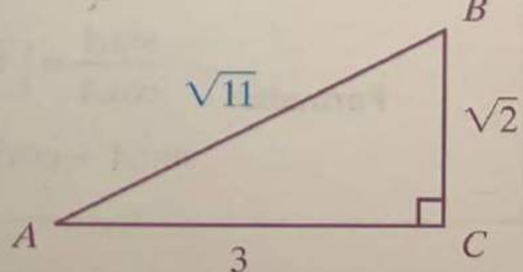


3. Given $\tan A = \frac{\sqrt{2}}{3}$, find the values of $\sin A$ and $\cos A$.

[Sol] $AB = \sqrt{(\sqrt{2})^2 + 3^2} = \sqrt{11}$

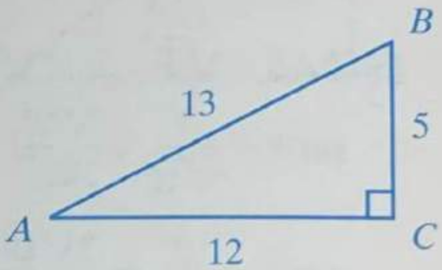
$$\sin A = \frac{\sqrt{2}}{\sqrt{11}} = \frac{\sqrt{22}}{11}$$

$$\cos A = \frac{3}{\sqrt{11}} = \frac{3\sqrt{11}}{11}$$



4. Given $\sin A = \frac{5}{13}$, evaluate the following expressions.

(1) $\cos A = \sin(\boxed{90^\circ} - A) = \frac{12}{13}$ $\quad [AC = \sqrt{(13)^2 - 5^2} = 12]$



(2) $\cos A - \sin A = \frac{12}{13} - \frac{5}{13} = \frac{7}{13}$

(3) $\cos(90^\circ - A) = \sin A = \frac{5}{13}$

(4) $\frac{\sin A}{\cos A} = \frac{\frac{5}{13}}{\frac{12}{13}} = \frac{5}{13} \cdot \frac{13}{12} = \frac{5}{12}$

(5) $\tan A = \frac{5}{12}$

(6) $\tan(90^\circ - A) = \frac{1}{\tan A} = \frac{12}{5}$

(7) $\sin^2 A + \cos^2 A = \left(\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2 = \frac{25}{169} + \frac{144}{169} = \frac{169}{169} = 1$

Complete and check the Formulas.

Formulas $\quad \frac{\sin A}{\cos A} = \boxed{\tan} A$ $\quad$ From exercises (4) and (5)

$\sin^2 A + \cos^2 A = \boxed{1}$ $\quad$ From exercise (7)

Answers: tan, 1

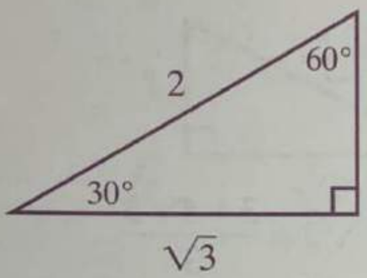
Notation: $\sin^2 A = (\sin A)^2$

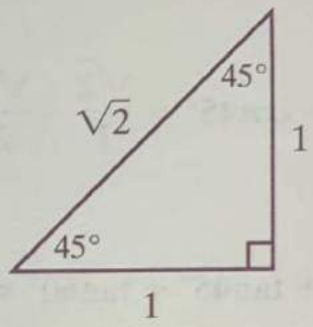
Trigonometric Functions 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	1	2~3	4~5	6~

1. Using the figures below, find the value of each of the trigonometric functions of 30°, 45° and 60°.





(1) $\sin 30° = \frac{1}{2}$ (2) $\sin 45° = \frac{\sqrt{2}}{2}$ (3) $\sin 60° = \frac{\sqrt{3}}{2}$

$\cos 30° = \frac{\sqrt{3}}{2}$ $\cos 45° = \frac{\sqrt{2}}{2}$ $\cos 60° = \frac{1}{2}$

$\tan 30° = \frac{\sqrt{3}}{3}$ $\tan 45° = 1$ $\tan 60° = \sqrt{3}$

2. Evaluate the following expressions as shown in the example.

Ex. $\boxed{\sin 30° \cos 60° = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}}$

(1) $\sin 45° \tan 45° = \frac{\sqrt{2}}{2} \cdot 1 = \frac{\sqrt{2}}{2}$

(2) $\sin 30° + \sin 60° = \frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2}$

(3) $\frac{\cos 45°}{\cos 30°} = \frac{\sqrt{2}}{2} \cdot \frac{2}{\sqrt{3}} = \frac{\sqrt{6}}{3}$

(4) $\tan 60° \tan 30° = \sqrt{3} \cdot \frac{\sqrt{3}}{3} = 1$

M 9 b

(5) $\sin 30° + \sin 45° + \sin 60° = \dfrac{1}{2} + \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{3}}{2} = \dfrac{1+\sqrt{2}+\sqrt{3}}{2}$

(6) $\cos 30° + \cos 45° + \cos 60° = \dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{2}}{2} + \dfrac{1}{2} = \dfrac{1+\sqrt{2}+\sqrt{3}}{2}$

(7) $\sin 45° - \cos 45° = \dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2} = 0$

(8) $\tan 30° + \tan 45° + \tan 60° = \dfrac{\sqrt{3}}{3} + 1 + \sqrt{3} = \dfrac{3+4\sqrt{3}}{3}$

(9) $\sin 30° \tan 45° + \cos 45° \tan 45° = \dfrac{1}{2} \cdot 1 + \dfrac{\sqrt{2}}{2} \cdot 1 = \dfrac{1+\sqrt{2}}{2}$

(10) $\dfrac{1}{\sin 30°} - \dfrac{1}{\cos 60°} \cdot \dfrac{1}{\tan 45°} = 2 - 2 = 0$

(11) $\dfrac{1}{\tan 30°} + \tan 45° = \sqrt{3} + 1$

(12) $\dfrac{\sin 60°}{\cos 30°} + \dfrac{\sin 45°}{\cos 45°} = 1 + 1 = 2$

(13) $\dfrac{1}{1+\sin 45°} = \dfrac{1}{1+\dfrac{\sqrt{2}}{2}} = \dfrac{1}{\dfrac{2+\sqrt{2}}{2}} = \dfrac{2}{2+\sqrt{2}} = \dfrac{4-2\sqrt{2}}{2} = 2-\sqrt{2}$

(14) $\dfrac{1+\tan 30°}{1-\tan 30°} = \dfrac{1+\dfrac{\sqrt{3}}{3}}{1-\dfrac{\sqrt{3}}{3}} = \dfrac{\dfrac{3+\sqrt{3}}{3}}{\dfrac{3-\sqrt{3}}{3}} = \dfrac{3+\sqrt{3}}{3-\sqrt{3}} = \dfrac{12+6\sqrt{3}}{6} = 2+\sqrt{3}$

KUMON

M 10 a

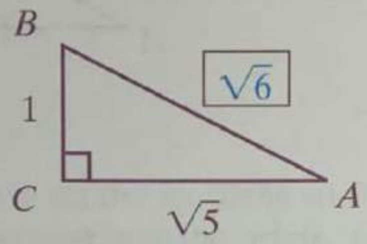
Trigonometric Functions 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	1	2	3	4~

1. Find the sin, cos and tan of the indicated angles in each of the following exercises.

(1)

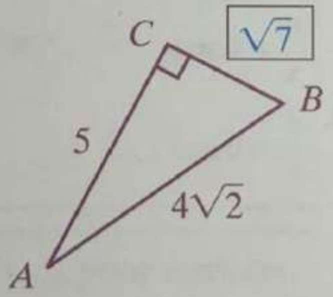


$\sin A = \frac{\sqrt{6}}{6}$ $\sin B = \frac{\sqrt{30}}{6}$

$\cos A = \frac{\sqrt{30}}{6}$ $\cos B = \frac{\sqrt{6}}{6}$

$\tan A = \frac{\sqrt{5}}{5}$ $\tan B = \sqrt{5}$

(2)

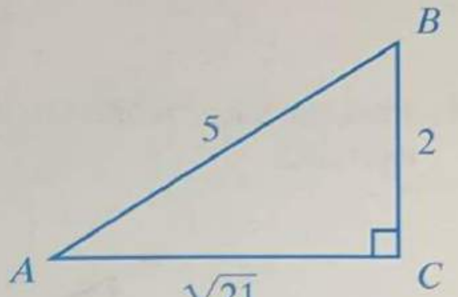


$\sin A = \frac{\sqrt{7}}{4\sqrt{2}} = \frac{\sqrt{14}}{8}$ $\sin B = \frac{5}{4\sqrt{2}} = \frac{5\sqrt{2}}{8}$

$\cos A = \frac{5}{4\sqrt{2}} = \frac{5\sqrt{2}}{8}$ $\cos B = \frac{\sqrt{7}}{4\sqrt{2}} = \frac{\sqrt{14}}{8}$

$\tan A = \frac{\sqrt{7}}{5}$ $\tan B = \frac{5}{\sqrt{7}} = \frac{5\sqrt{7}}{7}$

2. Given $\sin A = \frac{2}{5}$, evaluate the following expressions.



(1) $\cos A = \frac{\sqrt{21}}{5}$

(2) $\tan A = \frac{2}{\sqrt{21}} = \frac{2\sqrt{21}}{21}$

(3) $\sin(90° - A) = \cos A = \frac{\sqrt{21}}{5}$

(4) $\cos(90° - A) = \sin A = \frac{2}{5}$

(5) $\tan(90° - A) = \frac{1}{\tan A} = \frac{\sqrt{21}}{2}$

(6) $\frac{\sin A}{\cos A} = \tan A = \frac{2\sqrt{21}}{21}$

(7) $\sin^2 A + \cos^2 A = 1$

3. Evaluate the following expressions.

(1) $\sin 60° \tan 45° = \frac{\sqrt{3}}{2} \cdot 1 = \frac{\sqrt{3}}{2}$

(2) $\frac{\cos 30°}{\sin 30°} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$

(3) $\sin 45° + \cos 45° = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$

(4) $\frac{\tan 30°}{\tan 60°} + \cos 60° \sin 30° = \frac{\frac{\sqrt{3}}{3}}{\sqrt{3}} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$

KUMON

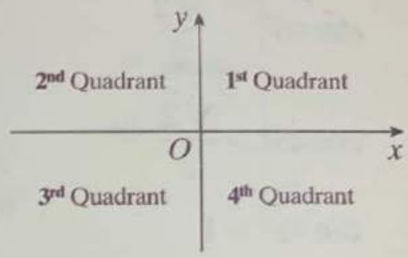
M 11 a

Trigonometric Functions 2

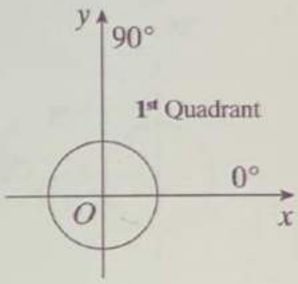
Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1~

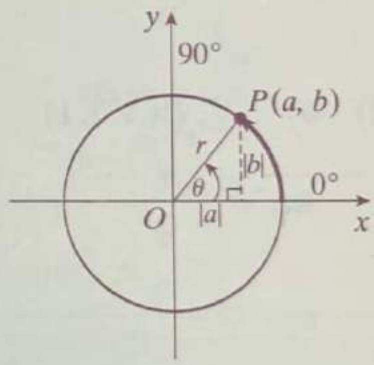
The axes of the xy-plane form four regions, called *Quadrants*, numbered in a counterclockwise order as shown in the figure on the right.



Let's place a circle on the xy-plane such that the circle's center is at the origin, O. Since a circle measures 360°, we can label the **1st Quadrant** to be from 0° to 90°.



Letting r be the radius of the circle and $P(a, b)$ be a point on the circle, and letting θ be the shortest angle formed from the x-axis to OP,



$$\sin\theta = \frac{b}{r}$$

When $0° < \theta < 90°$, $\quad \cos\theta = \frac{a}{r}$

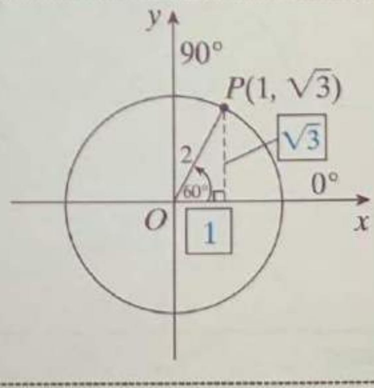
$$\tan\theta = \frac{b}{a}$$

Complete and check your answers.

Given $\theta = 60°$, $\quad \sin 60° = \frac{\sqrt{3}}{2}$

$$\cos 60° = \frac{1}{2}$$

$$\tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}$$



Answers: functions: $\frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{3}$; graph: 1, $\sqrt{3}$

Note: In this set, $|a|$ and $|b|$ are used to label the lengths of the legs of a right triangle.

M 11 b

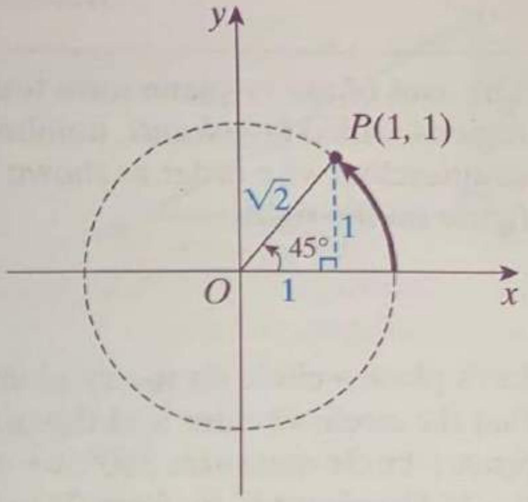
In each exercise, draw the diagram, and use it to evaluate the trigonometric functions of the given angle.

(1) $\theta = 45°$, $P(1,1)$

$$\sin 45° = \frac{\sqrt{2}}{2}$$

$$\cos 45° = \frac{\sqrt{2}}{2}$$

$$\tan 45° = 1$$

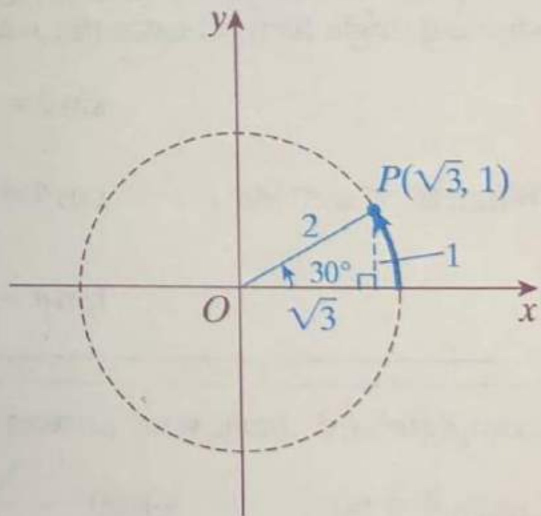


(2) $\theta = 30°$, $P(\sqrt{3},1)$

$$\sin 30° = \frac{1}{2}$$

$$\cos 30° = \frac{\sqrt{3}}{2}$$

$$\tan 30° = \frac{\sqrt{3}}{3}$$



Note:

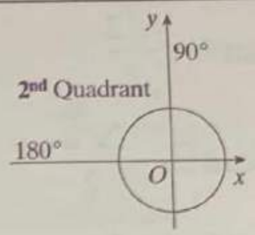
In the 1[st] Quadrant, both of the coordinates of point $P(a,b)$ have a positive value, i.e. $a>0$ and $b>0$. The radius of the circle, r, which in these exercises forms the hypotenuse of the right triangle, is always positive, i.e. $r>0$.

M 12 a Trigonometric Functions 2

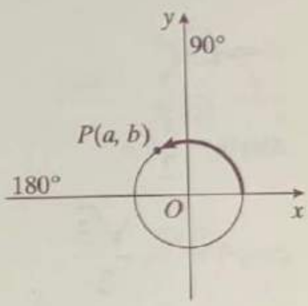
Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1~

If the 1st Quadrant measures from 0° to 90°, we can label the **2nd Quadrant** to be from 90° to 180°.



Given an angle θ, where $90° < \theta < 180°$, we can draw an arc, to mark the measure from 0° to $\theta°$ ending at point $P(a,b)$.

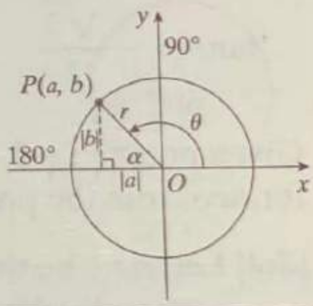


Letting α be the shortest angle formed from the x-axis to OP,
To determine the trigonometric values of θ, we draw a right triangle, using angle α.

$$\sin\theta = \sin\alpha = \frac{b}{r}$$

When $90° < \theta < 180°$, $\quad \cos\theta = -\cos\alpha = \frac{a}{r}$

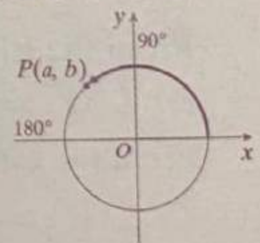
$$\tan\theta = -\tan\alpha = \frac{b}{a}$$



*Note that in the 2nd Quadrant, $a<0$ and $b>0$.

Ex. Given $\theta = 120°$, determine the value of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

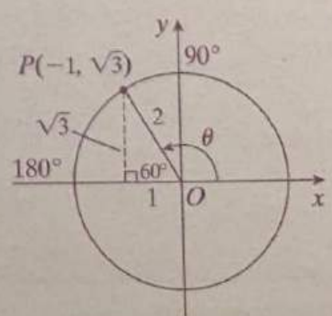
[Sol] Sketching the location of θ,
From the graph, $\alpha = 60°$.
Since we are working in the 2nd Quadrant,
$a = -1$ and $b = \sqrt{3}$



Therefore, $\quad \sin\theta = \frac{\sqrt{3}}{2}$

$$\cos\theta = \boxed{-\frac{1}{2}}$$

$$\tan\theta = \boxed{-\sqrt{3}}$$

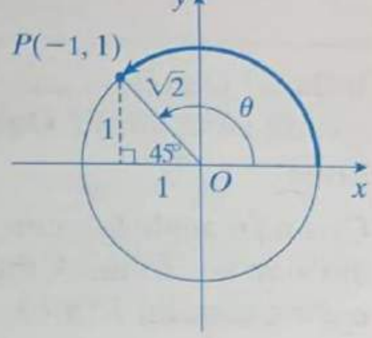


Answers: $-\frac{1}{2}$, $-\sqrt{3}$

M 12 b

1. In each exercise, draw the diagram, and use it to evaluate the trigonometric functions of the given angle.

(1) $\theta = 135°$

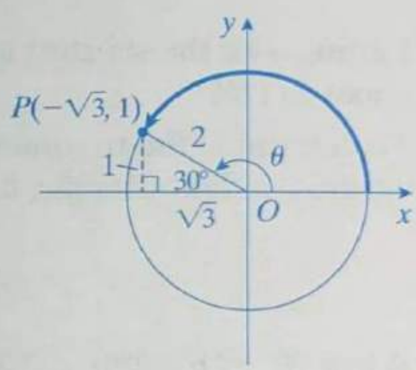


$\sin\theta = \dfrac{\sqrt{2}}{2}$

$\cos\theta = -\dfrac{\sqrt{2}}{2}$

$\tan\theta = -1$

(2) $\theta = 150°$



$\sin\theta = \dfrac{1}{2}$

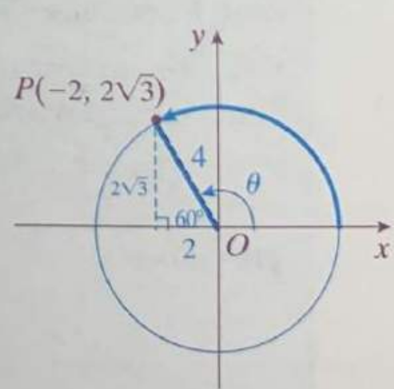
$\cos\theta = -\dfrac{\sqrt{3}}{2}$

$\tan\theta = -\dfrac{\sqrt{3}}{3}$

2. Given point $P(-2, 2\sqrt{3})$, find the value of the sin, cos and tan of angle θ, formed from the positive x-axis (moving counterclockwise) to OP.

[Sol] Letting θ be the angle formed from the positive x-axis (moving counterclockwise) to OP, and letting α be the shortest angle formed from the x-axis to OP,

From the triangle,
$OP = \sqrt{(-2)^2 + (2\sqrt{3})^2} = 4$
From the ratio of the sides, $\alpha = 60°$



$\sin\theta = \dfrac{\sqrt{3}}{2}$

$\cos\theta = -\dfrac{1}{2}$

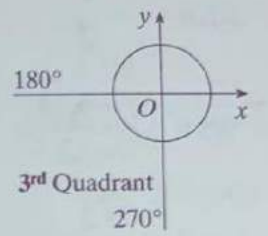
$\tan\theta = -\sqrt{3}$

Trigonometric Functions 2

Time : to : Date Name

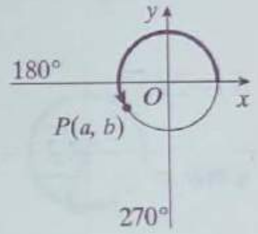
100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1~

Let's label the **3rd Quadrant** to be from 180° to 270°.



Given an angle θ, where $180° < \theta < 270°$, we can draw an arc, to mark the measure from $0°$ to $\theta°$ ending at point $P(a, b)$.

Letting α be the shortest angle formed from the x-axis to OP,

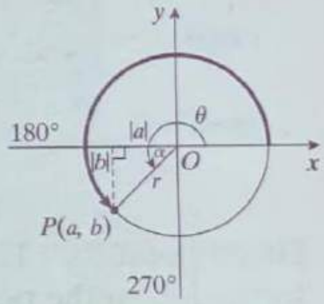


To determine the trigonometric values of θ, we draw a right triangle, using angle α.

$$\sin\theta = -\sin\alpha = \frac{b}{r}$$

When $180° < \theta < 270°$, $\quad \cos\theta = -\cos\alpha = \frac{a}{r}$

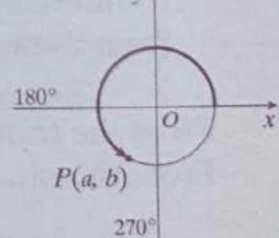
$$\tan\theta = \tan\alpha = \frac{b}{a}$$



*Note that in the 3rd Quadrant, $a<0$ and $b<0$.

Ex. Given $\theta = 240°$ determine the value of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] Sketching the location of θ,



From the graph, $\alpha = \boxed{60}°$.

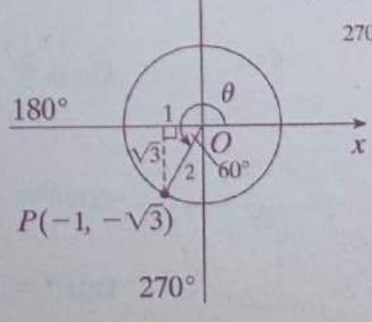
Since we are working in the 3rd Quadrant,

$a = \boxed{-1}$ and $b = \boxed{-\sqrt{3}}$

Therefore, $\quad \sin\theta = -\frac{\sqrt{3}}{2}$

$$\cos\theta = \boxed{-\frac{1}{2}}$$

$$\tan\theta = \boxed{\sqrt{3}}$$

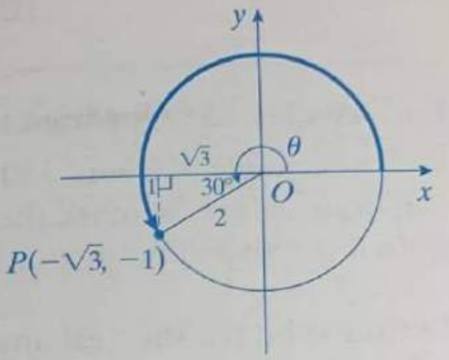


Answers: 60, -1, $-\sqrt{3}$, $-\frac{1}{2}$, $\sqrt{3}$

M 13 b.

1. In each exercise, draw the diagram, and use it to evaluate the trigonometric functions of the given angle.

(1) $\theta = 210°$

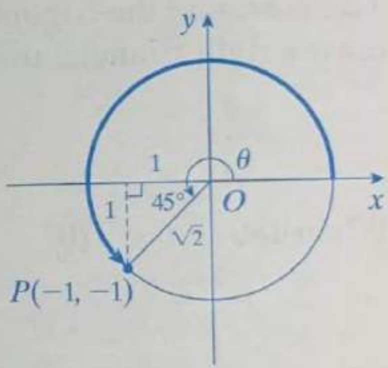


$$\sin\theta = -\frac{1}{2}$$

$$\cos\theta = -\frac{\sqrt{3}}{2}$$

$$\tan\theta = \frac{\sqrt{3}}{3}$$

(2) $\theta = 225°$



$$\sin\theta = -\frac{\sqrt{2}}{2}$$

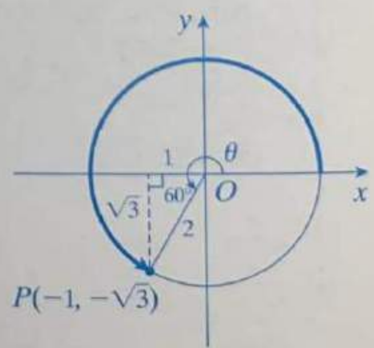
$$\cos\theta = -\frac{\sqrt{2}}{2}$$

$$\tan\theta = 1$$

2. Given point $P(-1, -\sqrt{3})$, find the value of the sin, cos and tan of angle θ, formed from the positive x-axis (moving counterclockwise) to OP.

[Sol] Letting θ be the angle formed from the positive x-axis (moving counterclockwise) to OP, and letting α be the shortest angle formed from the x-axis to OP,

From the triangle, $OP = 2$
From the ratio of the sides, $\alpha = 60°$



$$\sin\theta = -\frac{\sqrt{3}}{2}$$

$$\cos\theta = -\frac{1}{2}$$

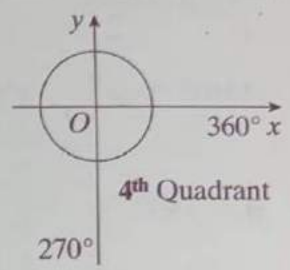
$$\tan\theta = \sqrt{3}$$

Trigonometric Functions 2

Time : to : Date Name

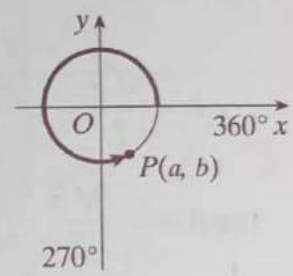
100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

We can label the **4th Quadrant** to be from 270° to 360°.



Given an angle θ, where $270° < \theta < 360°$, we can draw an arc, to mark the measure from 0° to $\theta°$ ending at point $P(a, b)$.

Letting α be the shortest angle formed from the x-axis to OP,

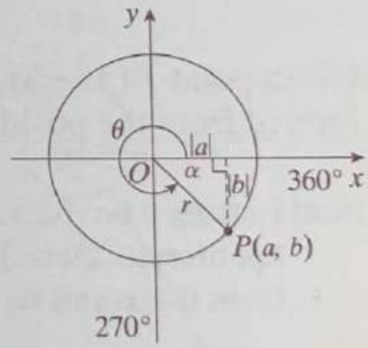


To determine the trigonometric values of θ, we draw a right triangle, using angle α.

$$\sin\theta = -\sin\alpha = \frac{b}{r}$$

When $270° < \theta < 360°$, $\cos\theta = \cos\alpha = \frac{a}{r}$

$$\tan\theta = -\tan\alpha = \frac{b}{a}$$



*Note that in the 4th Quadrant, $a>0$ and $b<0$.

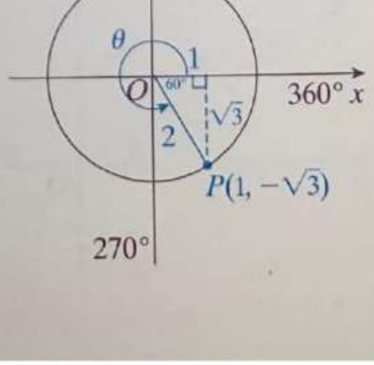
1. In each exercise, draw the diagram, and use it to evaluate the trigonometric functions of the given angle.

(1) $\theta = 300°$

$$\sin\theta = -\frac{\sqrt{3}}{2}$$

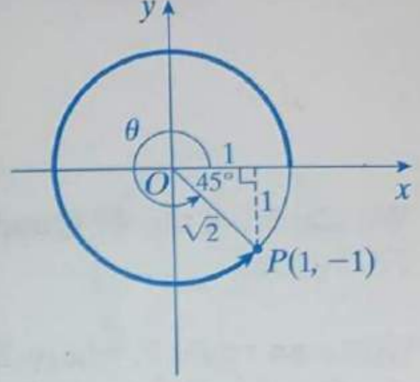
$$\cos\theta = \frac{1}{2}$$

$$\tan\theta = -\sqrt{3}$$



M 14 b

(2) $\theta = 315°$

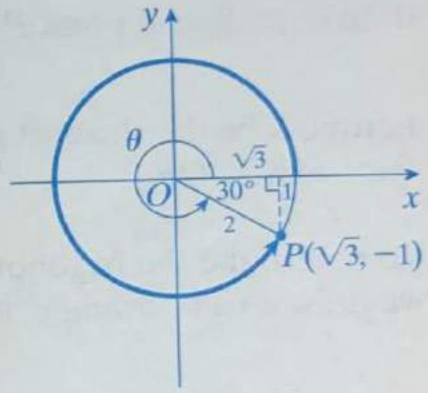


$\sin\theta = -\dfrac{\sqrt{2}}{2}$

$\cos\theta = \dfrac{\sqrt{2}}{2}$

$\tan\theta = -1$

(3) $\theta = 330°$



$\sin\theta = -\dfrac{1}{2}$

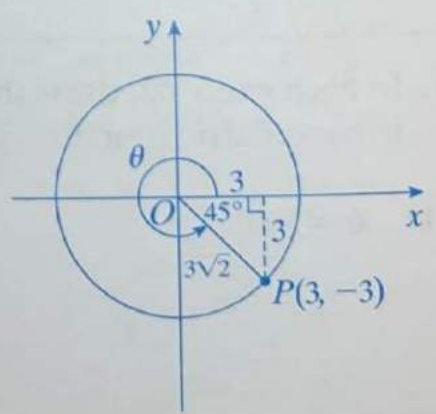
$\cos\theta = \dfrac{\sqrt{3}}{2}$

$\tan\theta = -\dfrac{\sqrt{3}}{3}$

2. Given point $P(3, -3)$, find the value of the sin, cos and tan of angle θ, formed from the positive x-axis (moving counterclockwise) to OP.

[Sol] Letting θ be the angle formed from the positive x-axis (moving counterclockwise) to OP, and letting α be the shortest angle formed from the x-axis to OP,

From the triangle, $OP = 3\sqrt{2}$
From the ratio of the sides, $\alpha = 45°$



$\sin\theta = -\dfrac{\sqrt{2}}{2}$

$\cos\theta = \dfrac{\sqrt{2}}{2}$

$\tan\theta = -1$

Trigonometric Functions 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

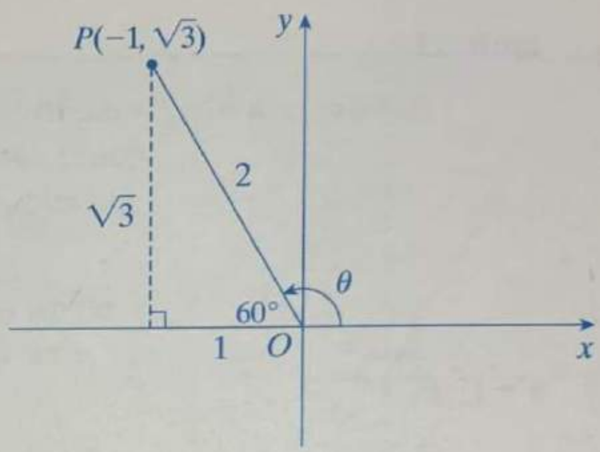
In each exercise, draw the diagram, and use it to evaluate the sin, cos and tan functions of the given angle.

(1) $\theta = 120°$

$\sin\theta = \dfrac{\sqrt{3}}{2}$

$\cos\theta = -\dfrac{1}{2}$

$\tan\theta = -\sqrt{3}$

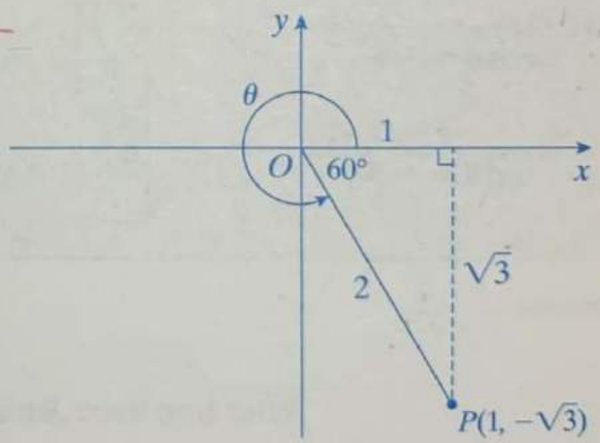


(2) $\theta = 300°$

$\sin\theta = -\dfrac{\sqrt{3}}{2}$

$\cos\theta = \dfrac{1}{2}$

$\tan\theta = -\sqrt{3}$

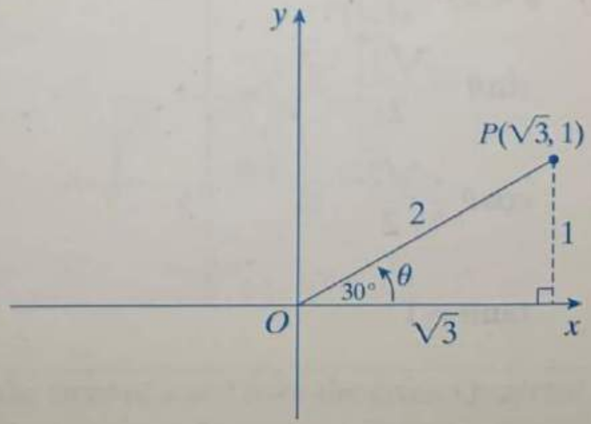


(3) $\theta = 30°$

$\sin\theta = \dfrac{1}{2}$

$\cos\theta = \dfrac{\sqrt{3}}{2}$

$\tan\theta = \dfrac{\sqrt{3}}{3}$



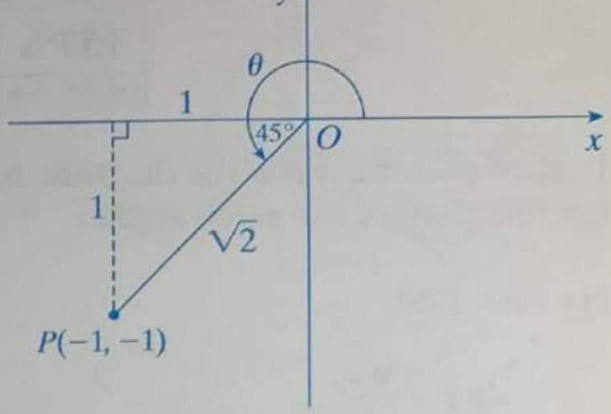
M 15 b

(4) $\theta = 225°$

$\sin\theta = -\dfrac{\sqrt{2}}{2}$

$\cos\theta = -\dfrac{\sqrt{2}}{2}$

$\tan\theta = 1$

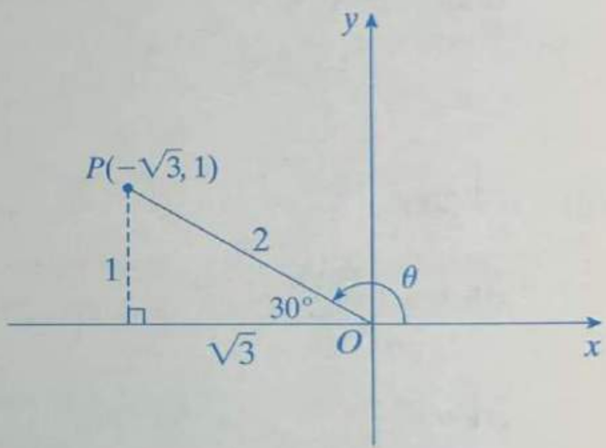


(5) $\theta = 150°$

$\sin\theta = \dfrac{1}{2}$

$\cos\theta = -\dfrac{\sqrt{3}}{2}$

$\tan\theta = -\dfrac{\sqrt{3}}{3}$

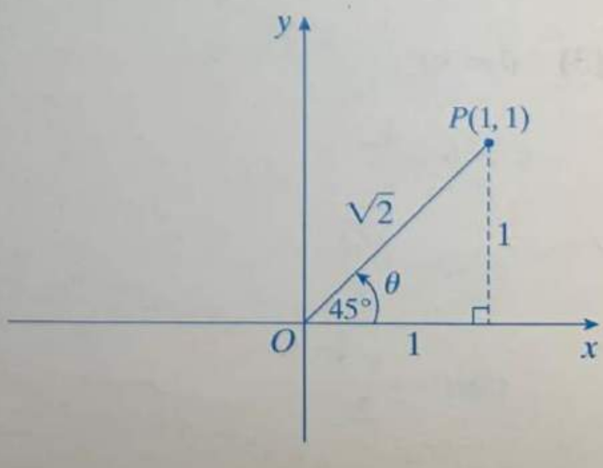


(6) $\theta = 45°$

$\sin\theta = \dfrac{\sqrt{2}}{2}$

$\cos\theta = \dfrac{\sqrt{2}}{2}$

$\tan\theta = 1$



Trigonometric Functions 2

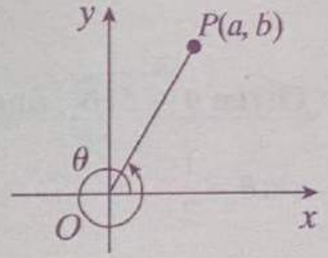
Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

Once we move one complete revolution, i.e. 360° around the circle, we can use the method studied in M11-15 to evaluate the trigonometric functions of angles greater than 360°.

Ex. Given $\theta = 390°$, find the value of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] As we measure θ, once we reach 360° we move 30° more, placing P in the 1st Quadrant.

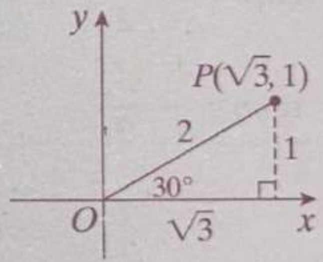


Letting α be the shortest angle formed by OP and the x-axis, $\alpha = 30°$

From the graph, $\sin\theta = \dfrac{1}{2}$

$\cos\theta = \boxed{\dfrac{\sqrt{3}}{2}}$

$\tan\theta = \boxed{\dfrac{\sqrt{3}}{3}}$



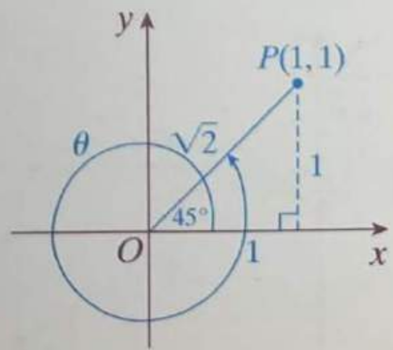
Answers: $\dfrac{\sqrt{3}}{2}$, $\dfrac{\sqrt{3}}{3}$

1. Given $\theta = 405°$, find the value of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

$\sin\theta = \dfrac{\sqrt{2}}{2}$

$\cos\theta = \dfrac{\sqrt{2}}{2}$

$\tan\theta = 1$



Reminder: In each exercise always check the signs of a and b for the given Quadrant.

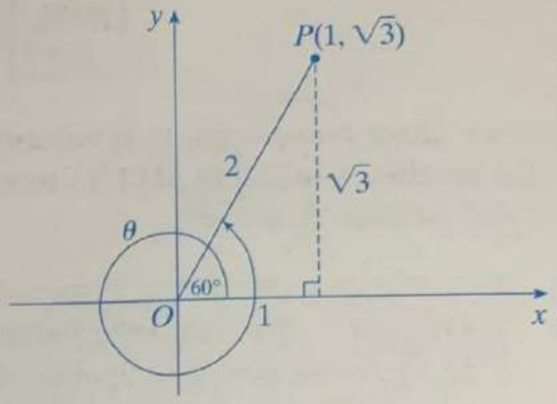
M 16 b

2. Given $\theta = 420°$, find the value of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

$\sin\theta = \frac{\sqrt{3}}{2}$

$\cos\theta = \frac{1}{2}$

$\tan\theta = \sqrt{3}$

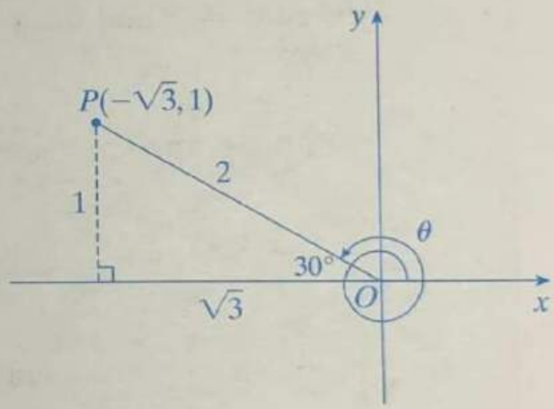


3. Given $\theta = 510°$, find the value of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

$\sin\theta = \frac{1}{2}$

$\cos\theta = -\frac{\sqrt{3}}{2}$

$\tan\theta = -\frac{\sqrt{3}}{3}$

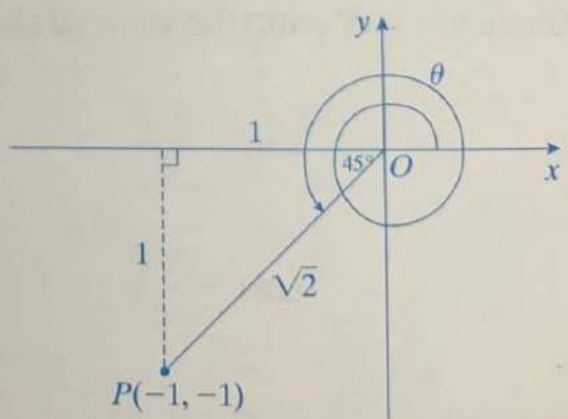


4. Given $\theta = 585°$, find the value of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

$\sin\theta = -\frac{\sqrt{2}}{2}$

$\cos\theta = -\frac{\sqrt{2}}{2}$

$\tan\theta = 1$



M 17 a Trigonometric Functions 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

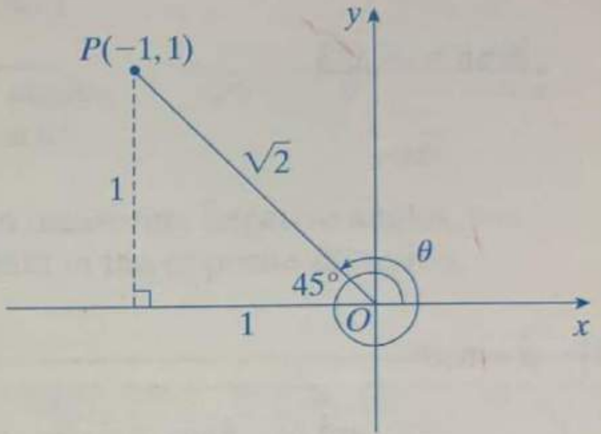
For each given angle θ, find the value of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

(1) $\theta = 495°$

$\sin\theta = \dfrac{\sqrt{2}}{2}$

$\cos\theta = -\dfrac{\sqrt{2}}{2}$

$\tan\theta = -1$

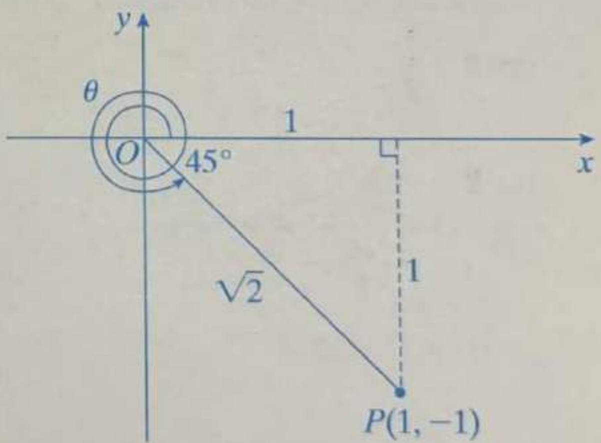


(2) $\theta = 675°$

$\sin\theta = -\dfrac{\sqrt{2}}{2}$

$\cos\theta = \dfrac{\sqrt{2}}{2}$

$\tan\theta = -1$

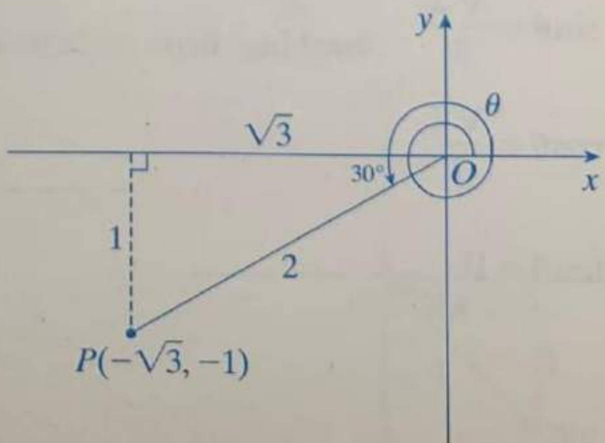


(3) $\theta = 570°$

$\sin\theta = -\dfrac{1}{2}$

$\cos\theta = -\dfrac{\sqrt{3}}{2}$

$\tan\theta = \dfrac{\sqrt{3}}{3}$



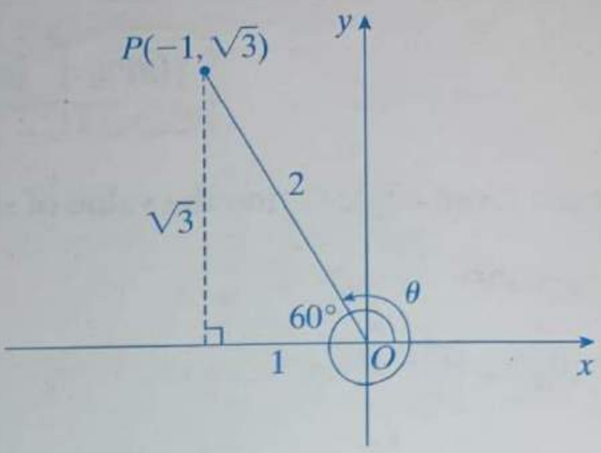
M 17 b

(4) $\theta = 480°$

$\sin\theta = \dfrac{\sqrt{3}}{2}$

$\cos\theta = -\dfrac{1}{2}$

$\tan\theta = -\sqrt{3}$

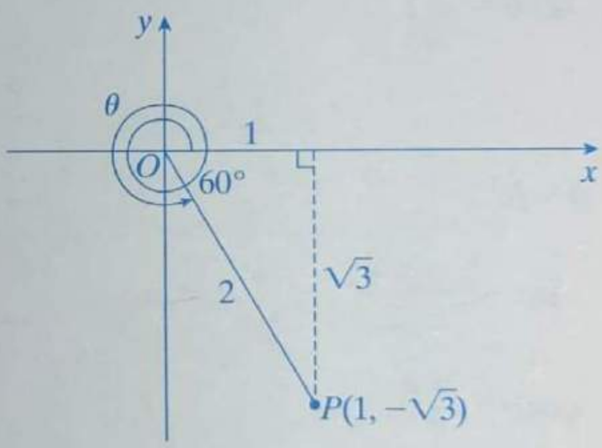


(5) $\theta = 660°$

$\sin\theta = -\dfrac{\sqrt{3}}{2}$

$\cos\theta = \dfrac{1}{2}$

$\tan\theta = -\sqrt{3}$

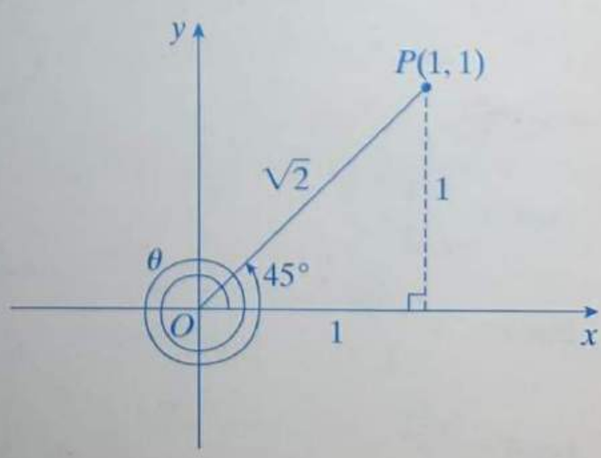


(6)* $\theta = 765°$

$\sin\theta = \dfrac{\sqrt{2}}{2}$

$\cos\theta = \dfrac{\sqrt{2}}{2}$

$\tan\theta = 1$

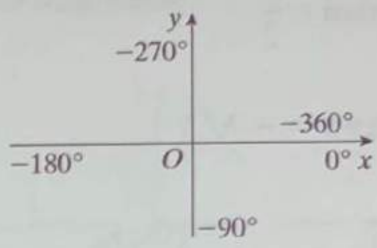


Trigonometric Functions 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

Up to this point we have been working with positive angles. We can use the method studied in M11-17 to evaluate the trigonometric functions of negative angles, i.e. angles whose measure is less than $0°$.



As shown on the figure above, when measuring negative angles, we measure the degrees of each Quadrant in the opposite direction, clockwise.

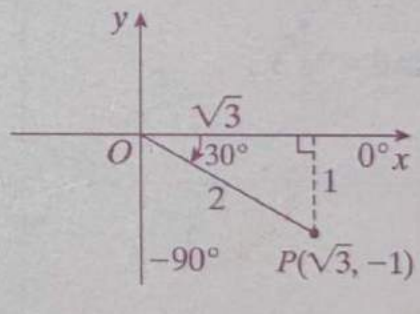
Ex. Given $\theta = -30°$, find the value of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

[Sol] We begin measuring from $0°$ towards the 4th Quadrant.
From the graph,

$$\sin\theta = -\frac{1}{2}$$

$$\cos\theta = \boxed{\frac{\sqrt{3}}{2}}$$

$$\tan\theta = \boxed{-\frac{\sqrt{3}}{3}}$$



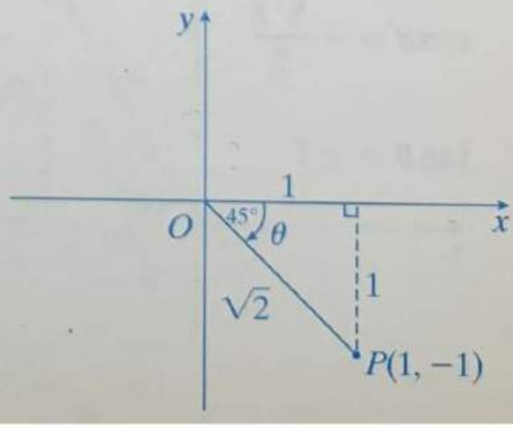
Answers: $\frac{\sqrt{3}}{2}$, $-\frac{\sqrt{3}}{3}$

1. Given $\theta = -45°$, find the value of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

$$\sin\theta = -\frac{\sqrt{2}}{2}$$

$$\cos\theta = \frac{\sqrt{2}}{2}$$

$$\tan\theta = -1$$

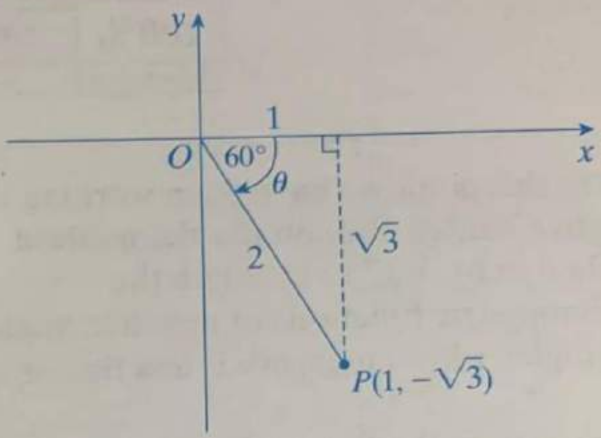


2. Given $\theta = -60°$, find the value of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

$\sin\theta = -\dfrac{\sqrt{3}}{2}$

$\cos\theta = \dfrac{1}{2}$

$\tan\theta = -\sqrt{3}$

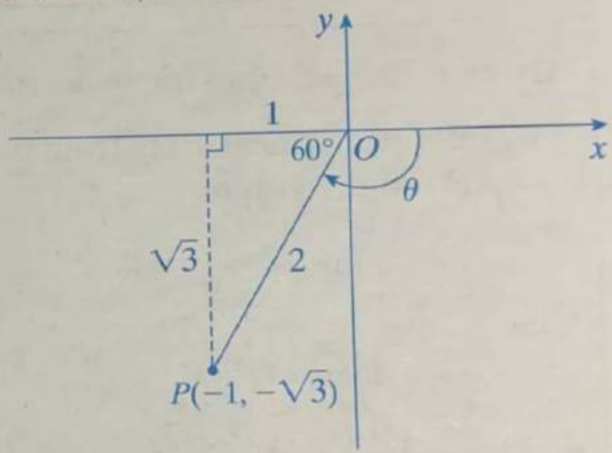


3. Given $\theta = -120°$, find the value of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

$\sin\theta = -\dfrac{\sqrt{3}}{2}$

$\cos\theta = -\dfrac{1}{2}$

$\tan\theta = \sqrt{3}$

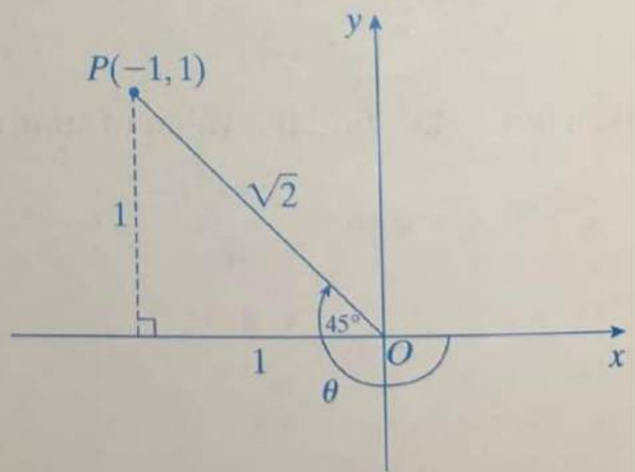


4. Given $\theta = -225°$, find the value of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

$\sin\theta = \dfrac{\sqrt{2}}{2}$

$\cos\theta = -\dfrac{\sqrt{2}}{2}$

$\tan\theta = -1$



KUMON

Trigonometric Functions 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

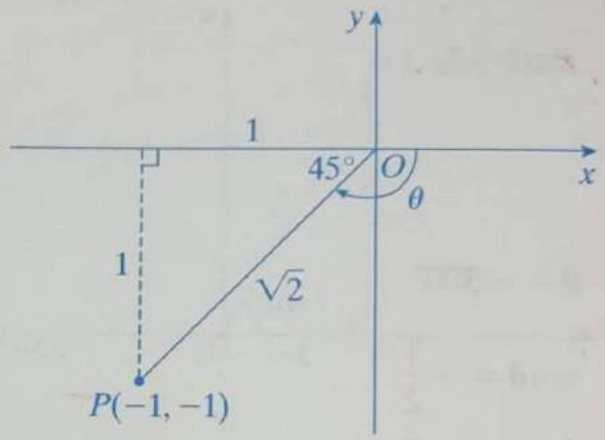
For each given angle θ find the value of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

(1) $\theta = -135°$

$\sin\theta = -\dfrac{\sqrt{2}}{2}$

$\cos\theta = -\dfrac{\sqrt{2}}{2}$

$\tan\theta = 1$

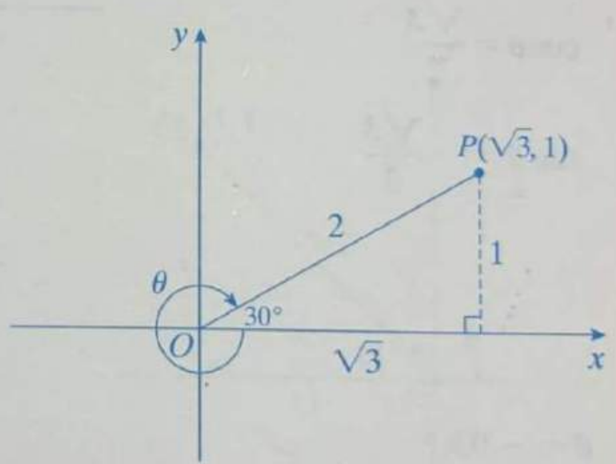


(2) $\theta = -330°$

$\sin\theta = \dfrac{1}{2}$

$\cos\theta = \dfrac{\sqrt{3}}{2}$

$\tan\theta = \dfrac{\sqrt{3}}{3}$

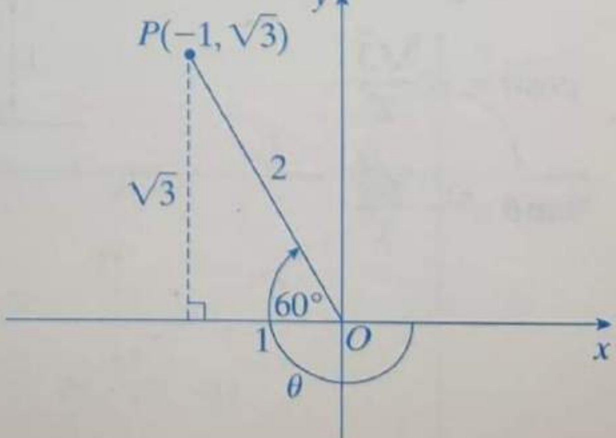


(3) $\theta = -240°$

$\sin\theta = \dfrac{\sqrt{3}}{2}$

$\cos\theta = -\dfrac{1}{2}$

$\tan\theta = -\sqrt{3}$

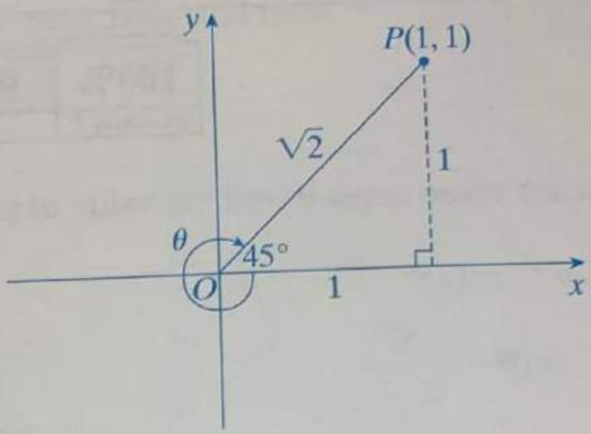


(4) $\theta = -315°$

$\sin\theta = \frac{\sqrt{2}}{2}$

$\cos\theta = \frac{\sqrt{2}}{2}$

$\tan\theta = 1$

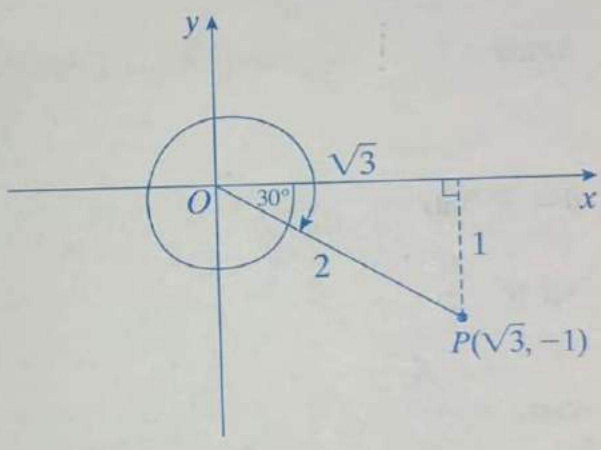


(5) $\theta = -390°$

$\sin\theta = -\frac{1}{2}$

$\cos\theta = \frac{\sqrt{3}}{2}$

$\tan\theta = -\frac{\sqrt{3}}{3}$

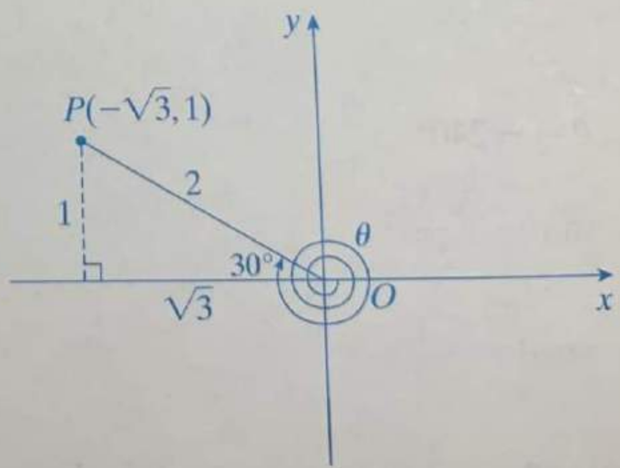


(6)* $\theta = -930°$

$\sin\theta = \frac{1}{2}$

$\cos\theta = -\frac{\sqrt{3}}{2}$

$\tan\theta = -\frac{\sqrt{3}}{3}$



M 20 a

Trigonometric Functions 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

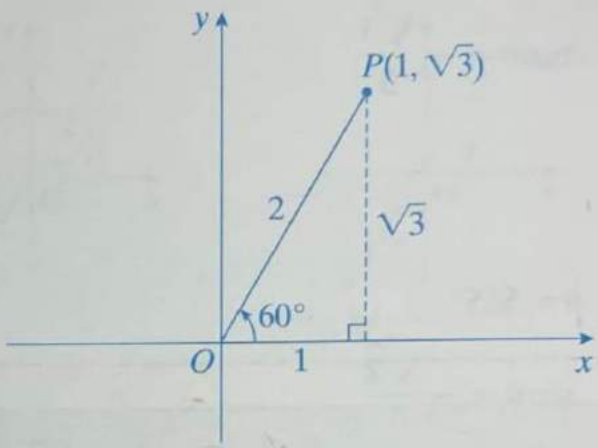
For each given angle θ find the value of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

(1) $\theta = 60°$

$\sin\theta = \dfrac{\sqrt{3}}{2}$

$\cos\theta = \dfrac{1}{2}$

$\tan\theta = \sqrt{3}$

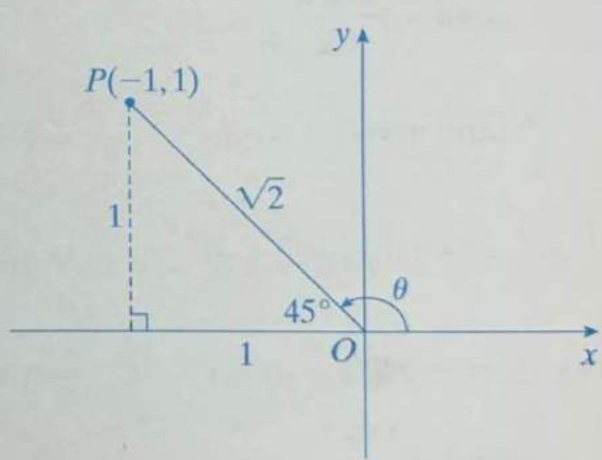


(2) $\theta = 135°$

$\sin\theta = \dfrac{\sqrt{2}}{2}$

$\cos\theta = -\dfrac{\sqrt{2}}{2}$

$\tan\theta = -1$

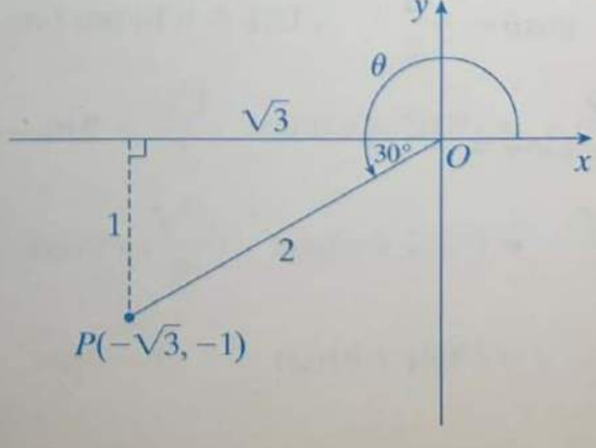


(3) $\theta = 210°$

$\sin\theta = -\dfrac{1}{2}$

$\cos\theta = -\dfrac{\sqrt{3}}{2}$

$\tan\theta = \dfrac{\sqrt{3}}{3}$



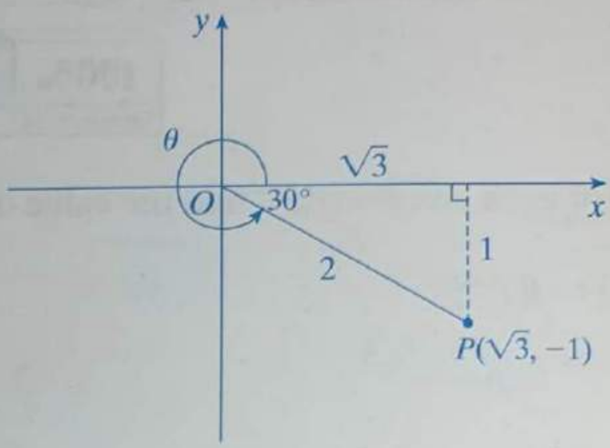
M 20 b

(4) $\theta = 330^\circ$

$\sin\theta = -\dfrac{1}{2}$

$\cos\theta = \dfrac{\sqrt{3}}{2}$

$\tan\theta = -\dfrac{\sqrt{3}}{3}$

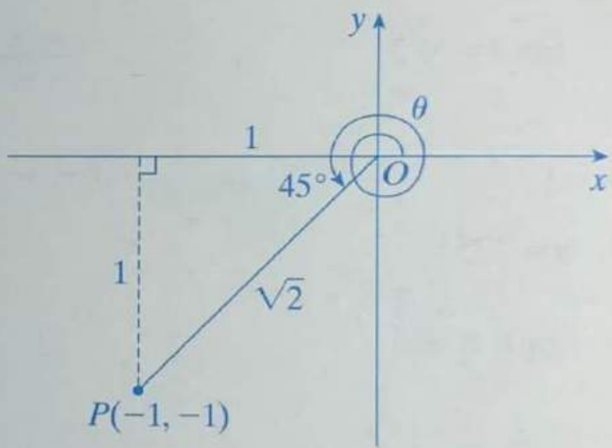


(5) $\theta = 585^\circ$

$\sin\theta = -\dfrac{\sqrt{2}}{2}$

$\cos\theta = -\dfrac{\sqrt{2}}{2}$

$\tan\theta = 1$

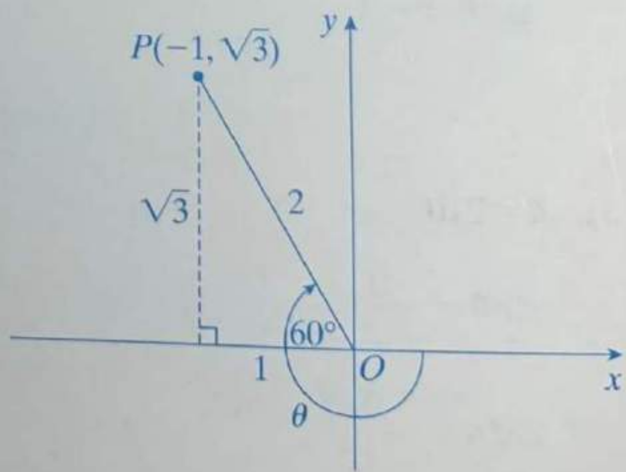


(6) $\theta = -240^\circ$

$\sin\theta = \dfrac{\sqrt{3}}{2}$

$\cos\theta = -\dfrac{1}{2}$

$\tan\theta = -\sqrt{3}$

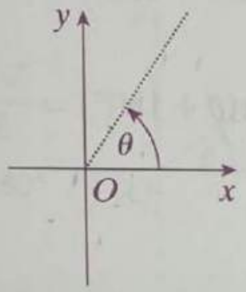


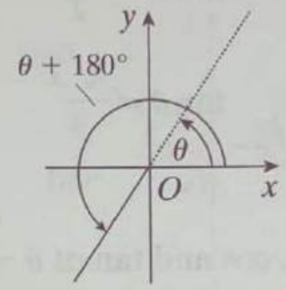
Trigonometric Functions 3

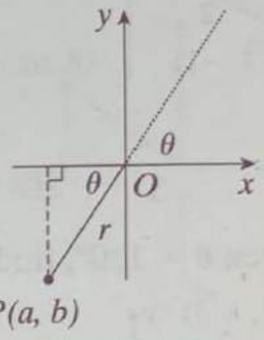
Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

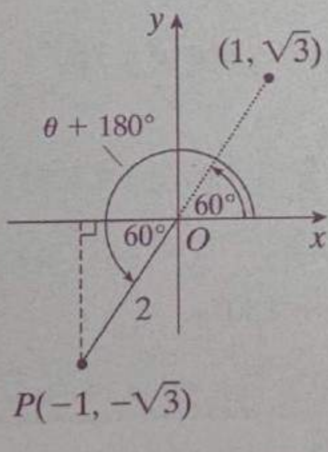
From θ, measuring 180° counterclockwise results to $\theta + 180°$.







Ex. Given $\theta = 60°$, find the sin, cos and tan of $\theta + 180°$.

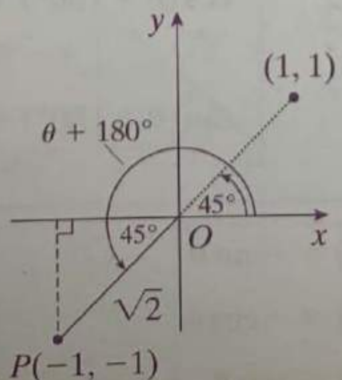


$\sin\theta = \dfrac{\sqrt{3}}{2}$ $\quad\sin(\theta + 180°) = -\dfrac{\sqrt{3}}{2}$

$\cos\theta = \dfrac{1}{2}$ $\quad\cos(\theta + 180°) = -\dfrac{1}{2}$

$\tan\theta = \sqrt{3}$ $\quad\tan(\theta + 180°) = \sqrt{3}$

1. Given $\theta = 45°$, find the sin, cos and tan of $\theta + 180°$.



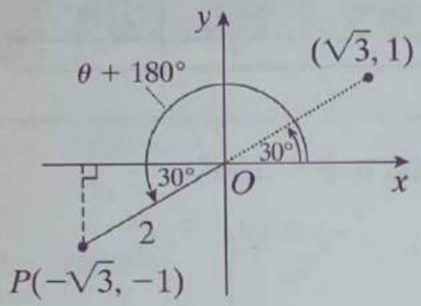
$\sin\theta = \dfrac{\sqrt{2}}{2}$ $\quad\sin(\theta + 180°) = -\dfrac{\sqrt{2}}{2}$

$\cos\theta = \dfrac{\sqrt{2}}{2}$ $\quad\cos(\theta + 180°) = -\dfrac{\sqrt{2}}{2}$

$\tan\theta = 1$ $\quad\tan(\theta + 180°) = 1$

M 21 b

2. Given $\theta = 30°$, find the sin, cos and tan of $\theta + 180°$.

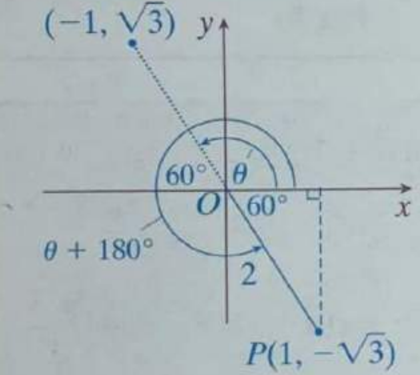


$\sin\theta = \frac{1}{2}$ $\quad$ $\sin(\theta + 180°) = -\frac{1}{2}$

$\cos\theta = \frac{\sqrt{3}}{2}$ $\quad$ $\cos(\theta + 180°) = -\frac{\sqrt{3}}{2}$

$\tan\theta = \frac{\sqrt{3}}{3}$ $\quad$ $\tan(\theta + 180°) = \frac{\sqrt{3}}{3}$

3. Given $\theta = 120°$, find the sin, cos and tan of $\theta + 180°$.

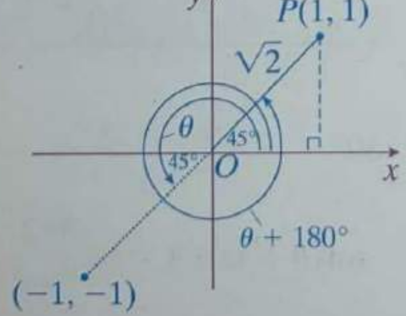


$\sin\theta = \frac{\sqrt{3}}{2}$ $\quad$ $\sin(\theta + 180°) = -\frac{\sqrt{3}}{2}$

$\cos\theta = -\frac{1}{2}$ $\quad$ $\cos(\theta + 180°) = \frac{1}{2}$

$\tan\theta = -\sqrt{3}$ $\quad$ $\tan(\theta + 180°) = -\sqrt{3}$

4. Given $\theta = 225°$, find the sin, cos and tan of $\theta + 180°$.



$\sin\theta = -\frac{\sqrt{2}}{2}$ $\quad$ $\sin(\theta + 180°) = \frac{\sqrt{2}}{2}$

$\cos\theta = -\frac{\sqrt{2}}{2}$ $\quad$ $\cos(\theta + 180°) = \frac{\sqrt{2}}{2}$

$\tan\theta = 1$ $\quad$ $\tan(\theta + 180°) = 1$

Formulas

$$\boxed{\begin{aligned} \sin(\theta + 180°) &= -\sin\theta \\ \cos(\theta + 180°) &= -\cos\theta \\ \tan(\theta + 180°) &= \tan\theta \end{aligned}}$$

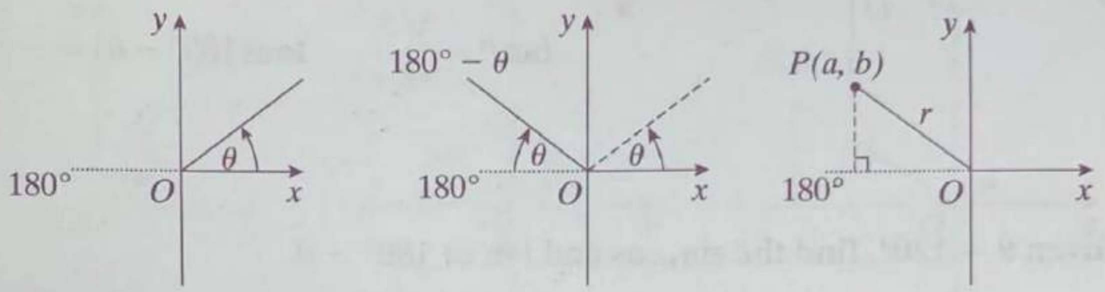
KUMON

M 22 a Trigonometric Functions 3

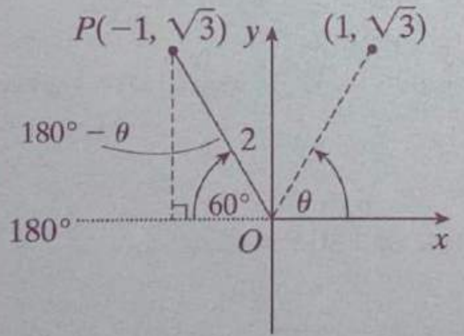
Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

From 180°, measuring θ degrees clockwise results to $180° - \theta$.



Ex. Given $\theta = 60°$, find the sin, cos and tan of $180° - \theta$.

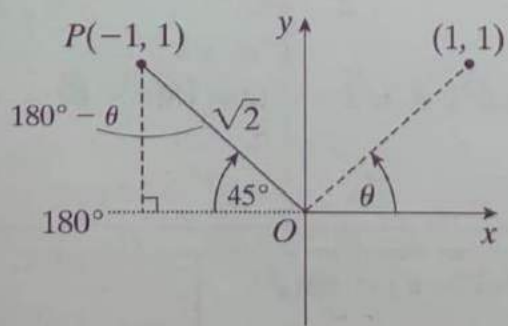


$\sin\theta = \frac{\sqrt{3}}{2}$ $\sin(180° - \theta) = \frac{\sqrt{3}}{2}$

$\cos\theta = \frac{1}{2}$ $\cos(180° - \theta) = -\frac{1}{2}$

$\tan\theta = \sqrt{3}$ $\tan(180° - \theta) = -\sqrt{3}$

1. Given $\theta = 45°$, find the sin, cos and tan of $180° - \theta$.



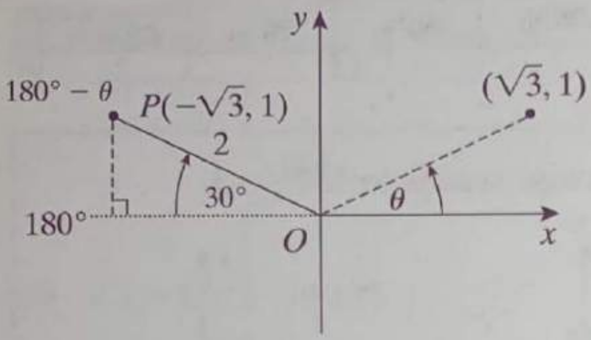
$\sin\theta = \frac{\sqrt{2}}{2}$ $\sin(180° - \theta) = \frac{\sqrt{2}}{2}$

$\cos\theta = \frac{\sqrt{2}}{2}$ $\cos(180° - \theta) = -\frac{\sqrt{2}}{2}$

$\tan\theta = 1$ $\tan(180° - \theta) = -1$

M 22 b

2. Given $\theta = 30°$, find the sin, cos and tan of $180° - \theta$.

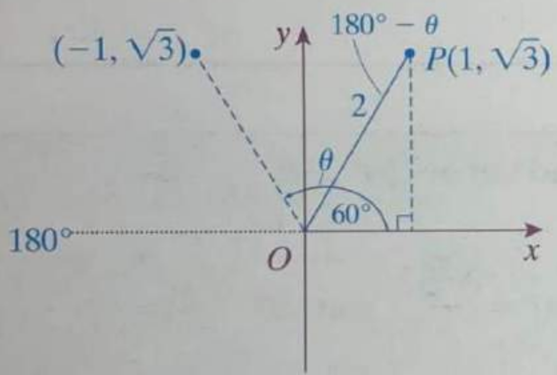


$\sin\theta = \frac{1}{2}$ $\quad \sin(180° - \theta) = \frac{1}{2}$

$\cos\theta = \frac{\sqrt{3}}{2}$ $\quad \cos(180° - \theta) = -\frac{\sqrt{3}}{2}$

$\tan\theta = \frac{\sqrt{3}}{3}$ $\quad \tan(180° - \theta) = -\frac{\sqrt{3}}{3}$

3. Given $\theta = 120°$, find the sin, cos and tan of $180° - \theta$.

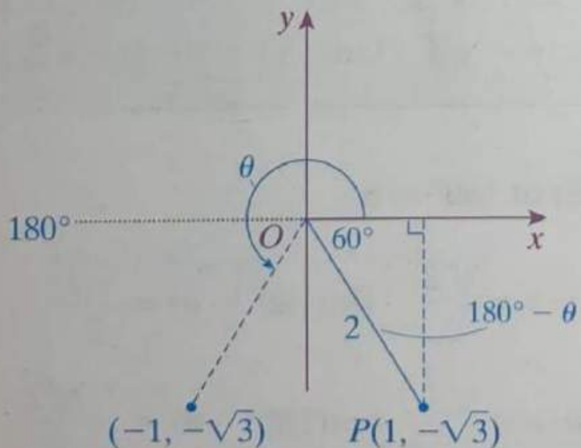


$\sin\theta = \frac{\sqrt{3}}{2}$ $\quad \sin(180° - \theta) = \frac{\sqrt{3}}{2}$

$\cos\theta = -\frac{1}{2}$ $\quad \cos(180° - \theta) = \frac{1}{2}$

$\tan\theta = -\sqrt{3}$ $\quad \tan(180° - \theta) = \sqrt{3}$

4. Given $\theta = 240°$, find the sin, cos and tan of $180° - \theta$.



$\sin\theta = -\frac{\sqrt{3}}{2}$ $\quad \sin(180° - \theta) = -\frac{\sqrt{3}}{2}$

$\cos\theta = -\frac{1}{2}$ $\quad \cos(180° - \theta) = \frac{1}{2}$

$\tan\theta = \sqrt{3}$ $\quad \tan(180° - \theta) = -\sqrt{3}$

Formulas

$$\boxed{\begin{aligned} \sin(180° - \theta) &= \sin\theta \\ \cos(180° - \theta) &= -\cos\theta \\ \tan(180° - \theta) &= -\tan\theta \end{aligned}}$$

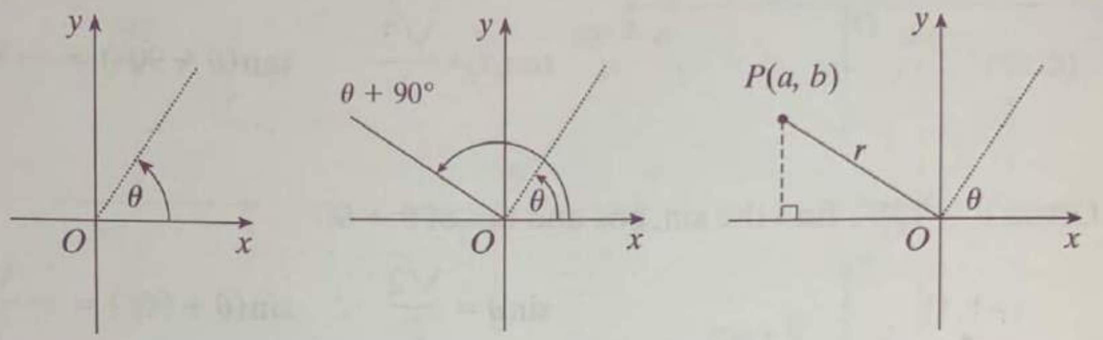
M 23 a

Trigonometric Functions 3

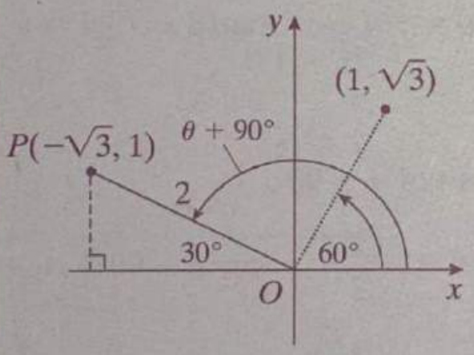
Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

From θ, measuring 90° counterclockwise results to $\theta + 90^\circ$.



Ex. Given $\theta = 60^\circ$, find the sin, cos and tan of $\theta + 90^\circ$.

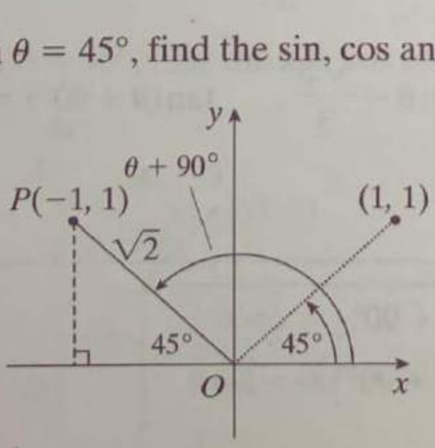


$\sin\theta = \dfrac{\sqrt{3}}{2}$ $\quad \sin(\theta + 90^\circ) = \dfrac{1}{2}$

$\cos\theta = \dfrac{1}{2}$ $\quad \cos(\theta + 90^\circ) = -\dfrac{\sqrt{3}}{2}$

$\tan\theta = \sqrt{3}$ $\quad \tan(\theta + 90^\circ) = -\dfrac{\sqrt{3}}{3}$

1. Given $\theta = 45^\circ$, find the sin, cos and tan of $\theta + 90^\circ$.



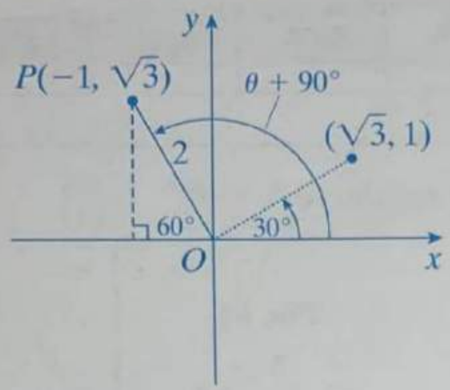
$\sin\theta = \dfrac{\sqrt{2}}{2}$ $\quad \sin(\theta + 90^\circ) = \dfrac{\sqrt{2}}{2}$

$\cos\theta = \dfrac{\sqrt{2}}{2}$ $\quad \cos(\theta + 90^\circ) = -\dfrac{\sqrt{2}}{2}$

$\tan\theta = 1$ $\quad \tan(\theta + 90^\circ) = -1$

M 23 b

2. Given $\theta = 30°$, find the sin, cos and tan of $\theta + 90°$.

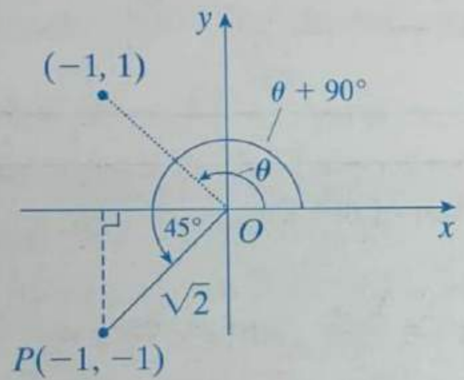


$\sin\theta = \frac{1}{2}$ $\quad$ $\sin(\theta + 90°) = \frac{\sqrt{3}}{2}$

$\cos\theta = \frac{\sqrt{3}}{2}$ $\quad$ $\cos(\theta + 90°) = -\frac{1}{2}$

$\tan\theta = \frac{\sqrt{3}}{3}$ $\quad$ $\tan(\theta + 90°) = -\sqrt{3}$

3. Given $\theta = 135°$, find the sin, cos and tan of $\theta + 90°$.

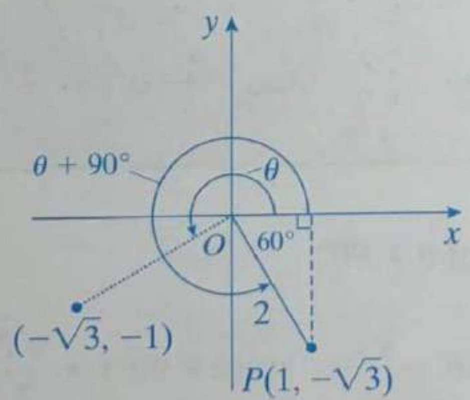


$\sin\theta = \frac{\sqrt{2}}{2}$ $\quad$ $\sin(\theta + 90°) = -\frac{\sqrt{2}}{2}$

$\cos\theta = -\frac{\sqrt{2}}{2}$ $\quad$ $\cos(\theta + 90°) = -\frac{\sqrt{2}}{2}$

$\tan\theta = -1$ $\quad$ $\tan(\theta + 90°) = 1$

4. Given $\theta = 210°$, find the sin, cos and tan of $\theta + 90°$.



$\sin\theta = -\frac{1}{2}$ $\quad$ $\sin(\theta + 90°) = -\frac{\sqrt{3}}{2}$

$\cos\theta = -\frac{\sqrt{3}}{2}$ $\quad$ $\cos(\theta + 90°) = \frac{1}{2}$

$\tan\theta = \frac{\sqrt{3}}{3}$ $\quad$ $\tan(\theta + 90°) = -\sqrt{3}$

Formulas

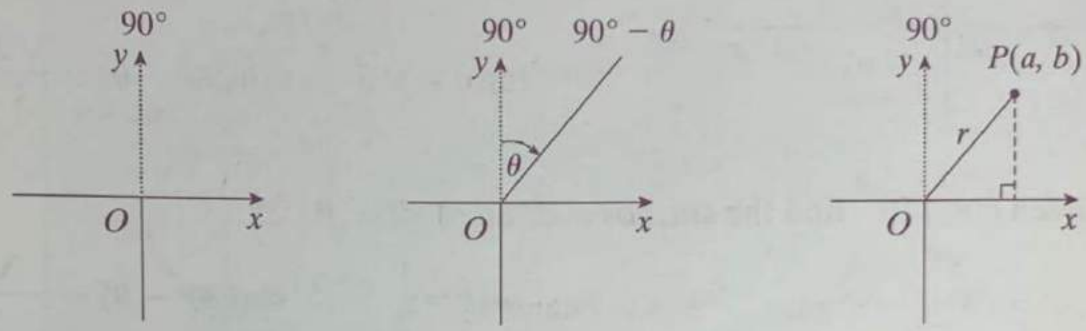
$$\boxed{\begin{aligned} \sin(\theta + 90°) &= \cos\theta \\ \cos(\theta + 90°) &= -\sin\theta \\ \tan(\theta + 90°) &= -\frac{1}{\tan\theta} \end{aligned}}$$

Trigonometric Functions 3

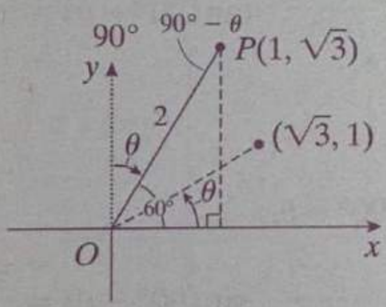
Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

From 90°, measuring θ degrees clockwise results to $90° - \theta$.



Ex. Given $\theta = 30°$, find the sin, cos and tan of $90° - \theta$.

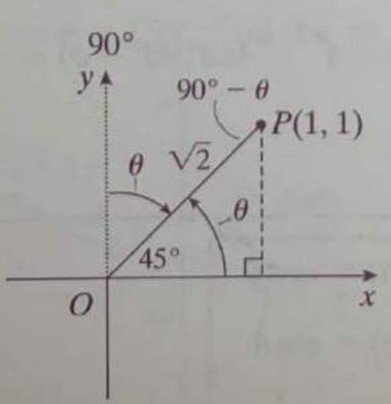


$\sin\theta = \frac{1}{2}$ $\quad \sin(90° - \theta) = \frac{\sqrt{3}}{2}$

$\cos\theta = \frac{\sqrt{3}}{2}$ $\quad \cos(90° - \theta) = \frac{1}{2}$

$\tan\theta = \frac{\sqrt{3}}{3}$ $\quad \tan(90° - \theta) = \sqrt{3}$

1. Given $\theta = 45°$, find the sin, cos and tan of $90° - \theta$.



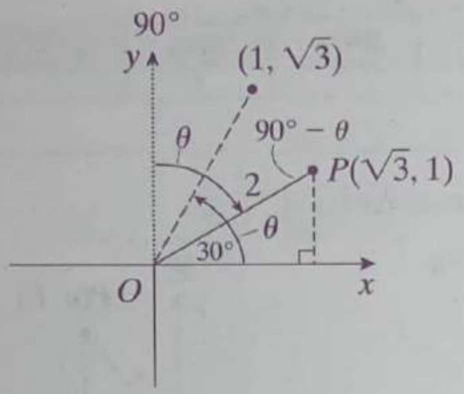
$\sin\theta = \frac{\sqrt{2}}{2}$ $\quad \sin(90° - \theta) = \frac{\sqrt{2}}{2}$

$\cos\theta = \frac{\sqrt{2}}{2}$ $\quad \cos(90° - \theta) = \frac{\sqrt{2}}{2}$

$\tan\theta = 1$ $\quad \tan(90° - \theta) = 1$

M 24 b

2. Given $\theta = 60°$, find the sin, cos and tan of $90° - \theta$.

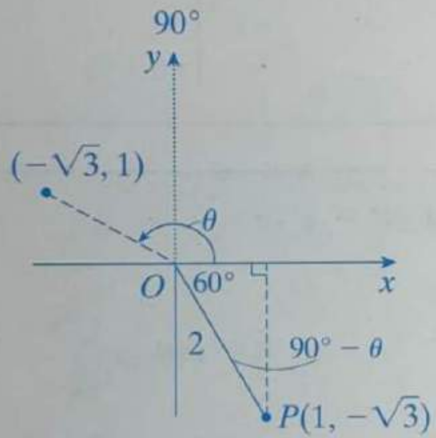


$\sin\theta = \frac{\sqrt{3}}{2}$ $\quad$ $\sin(90° - \theta) = \frac{1}{2}$

$\cos\theta = \frac{1}{2}$ $\quad$ $\cos(90° - \theta) = \frac{\sqrt{3}}{2}$

$\tan\theta = \sqrt{3}$ $\quad$ $\tan(90° - \theta) = \frac{\sqrt{3}}{3}$

3. Given $\theta = 150°$, find the sin, cos and tan of $90° - \theta$.

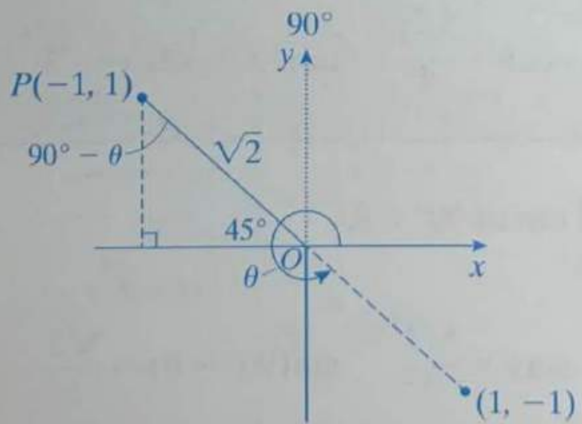


$\sin\theta = \frac{1}{2}$ $\quad$ $\sin(90° - \theta) = -\frac{\sqrt{3}}{2}$

$\cos\theta = -\frac{\sqrt{3}}{2}$ $\quad$ $\cos(90° - \theta) = \frac{1}{2}$

$\tan\theta = -\frac{\sqrt{3}}{3}$ $\quad$ $\tan(90° - \theta) = -\sqrt{3}$

4. Given $\theta = 315°$, find the sin, cos and tan of $90° - \theta$.



$\sin\theta = -\frac{\sqrt{2}}{2}$ $\quad$ $\sin(90° - \theta) = \frac{\sqrt{2}}{2}$

$\cos\theta = \frac{\sqrt{2}}{2}$ $\quad$ $\cos(90° - \theta) = -\frac{\sqrt{2}}{2}$

$\tan\theta = -1$ $\quad$ $\tan(90° - \theta) = -1$

Formulas

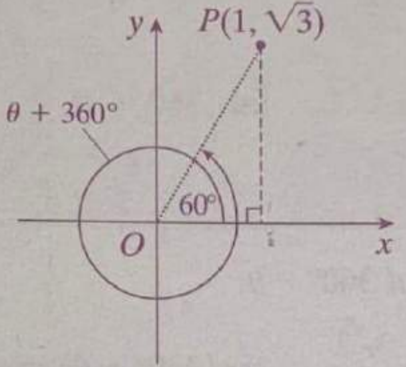
$$\boxed{\begin{aligned} \sin(90° - \theta) &= \cos\theta \\ \cos(90° - \theta) &= \sin\theta \\ \tan(90° - \theta) &= \frac{1}{\tan\theta} \end{aligned}}$$

Trigonometric Functions 3

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

Ex. Given $\theta = 60°$, find the sin, cos and tan of $\theta + 360°$.

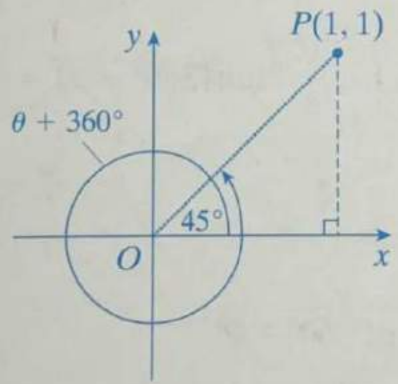


$\sin\theta = \dfrac{\sqrt{3}}{2}$ $\sin(\theta + 360°) = \dfrac{\sqrt{3}}{2}$

$\cos\theta = \dfrac{1}{2}$ $\cos(\theta + 360°) = \dfrac{1}{2}$

$\tan\theta = \sqrt{3}$ $\tan(\theta + 360°) = \sqrt{3}$

1. Given $\theta = 45°$, find the sin, cos and tan of $\theta + 360°$.

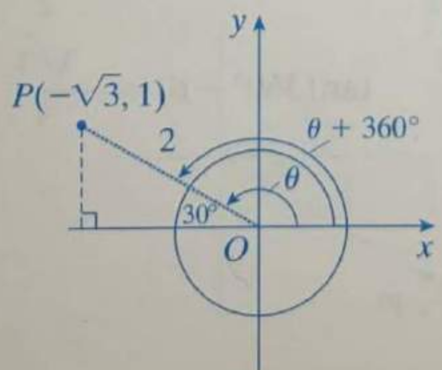


$\sin\theta = \dfrac{\sqrt{2}}{2}$ $\sin(\theta + 360°) = \dfrac{\sqrt{2}}{2}$

$\cos\theta = \dfrac{\sqrt{2}}{2}$ $\cos(\theta + 360°) = \dfrac{\sqrt{2}}{2}$

$\tan\theta = 1$ $\tan(\theta + 360°) = 1$

2. Given $\theta = 150°$, find the sin, cos and tan of $\theta + 360°$.



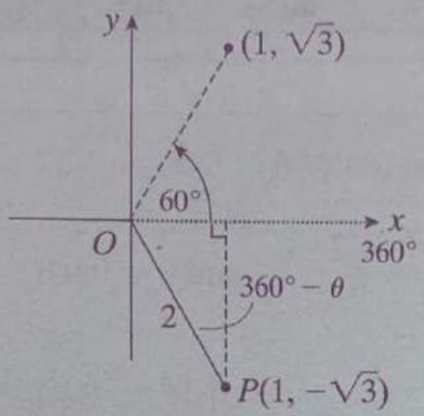
$\sin\theta = \dfrac{1}{2}$ $\sin(\theta + 360°) = \dfrac{1}{2}$

$\cos\theta = -\dfrac{\sqrt{3}}{2}$ $\cos(\theta + 360°) = -\dfrac{\sqrt{3}}{2}$

$\tan\theta = -\dfrac{\sqrt{3}}{3}$ $\tan(\theta + 360°) = -\dfrac{\sqrt{3}}{3}$

M 25 b

Ex. Given $\theta = 60°$, find the sin, cos and tan of $360° - \theta$.

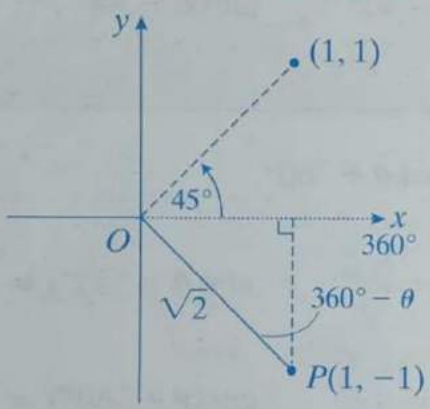


$\sin\theta = \dfrac{\sqrt{3}}{2}$ $\quad \sin(360° - \theta) = -\dfrac{\sqrt{3}}{2}$

$\cos\theta = \dfrac{1}{2}$ $\quad \cos(360° - \theta) = \dfrac{1}{2}$

$\tan\theta = \sqrt{3}$ $\quad \tan(360° - \theta) = -\sqrt{3}$

3. Given $\theta = 45°$, find the sin, cos and tan of $360° - \theta$.

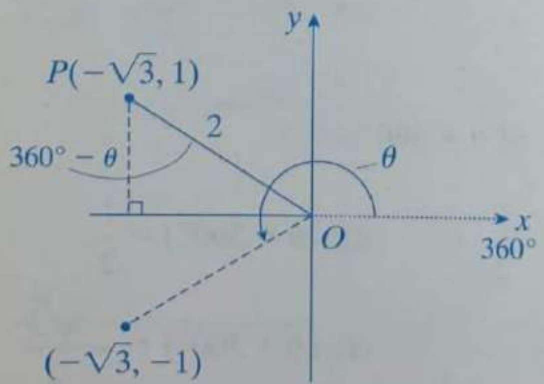


$\sin\theta = \dfrac{\sqrt{2}}{2}$ $\quad \sin(360° - \theta) = -\dfrac{\sqrt{2}}{2}$

$\cos\theta = \dfrac{\sqrt{2}}{2}$ $\quad \cos(360° - \theta) = \dfrac{\sqrt{2}}{2}$

$\tan\theta = 1$ $\quad \tan(360° - \theta) = -1$

4. Given $\theta = 210°$, find the sin, cos and tan of $360° - \theta$.



$\sin\theta = -\dfrac{1}{2}$ $\quad \sin(360° - \theta) = \dfrac{1}{2}$

$\cos\theta = -\dfrac{\sqrt{3}}{2}$ $\quad \cos(360° - \theta) = -\dfrac{\sqrt{3}}{2}$

$\tan\theta = \dfrac{\sqrt{3}}{3}$ $\quad \tan(360° - \theta) = -\dfrac{\sqrt{3}}{3}$

KUMON

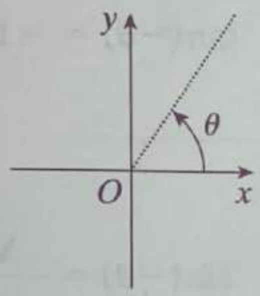
M 26 a

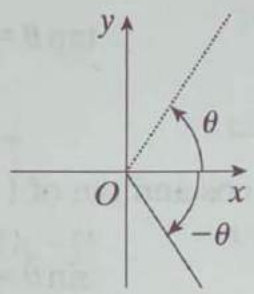
Trigonometric Functions 3

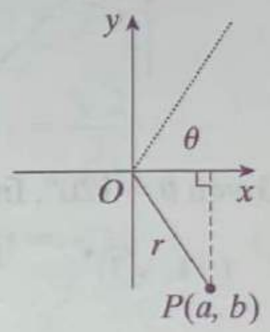
Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

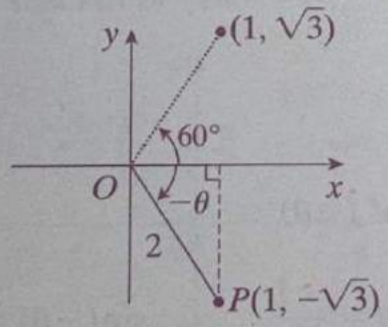
For every angle θ, there is an angle of equal measure in the opposite direction, $-\theta$.







Ex. Given $\theta = 60°$, find the sin, cos and tan of $(-\theta)$.

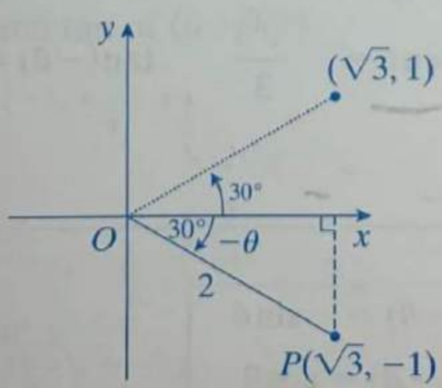


$\sin\theta = \frac{\sqrt{3}}{2}$ $\qquad$ $\sin(-\theta) = -\frac{\sqrt{3}}{2}$

$\cos\theta = \frac{1}{2}$ $\qquad$ $\cos(-\theta) = \frac{1}{2}$

$\tan\theta = \sqrt{3}$ $\qquad$ $\tan(-\theta) = -\sqrt{3}$

1. Given $\theta = 30°$, find the sin, cos and tan of $(-\theta)$.



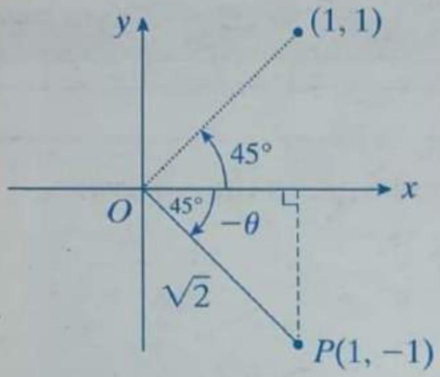
$\sin\theta = \frac{1}{2}$ $\qquad$ $\sin(-\theta) = -\frac{1}{2}$

$\cos\theta = \frac{\sqrt{3}}{2}$ $\qquad$ $\cos(-\theta) = \frac{\sqrt{3}}{2}$

$\tan\theta = \frac{\sqrt{3}}{3}$ $\qquad$ $\tan(-\theta) = -\frac{\sqrt{3}}{3}$

M 26 b

2. Given $\theta = 45°$, find the sin, cos and tan of $(-\theta)$.

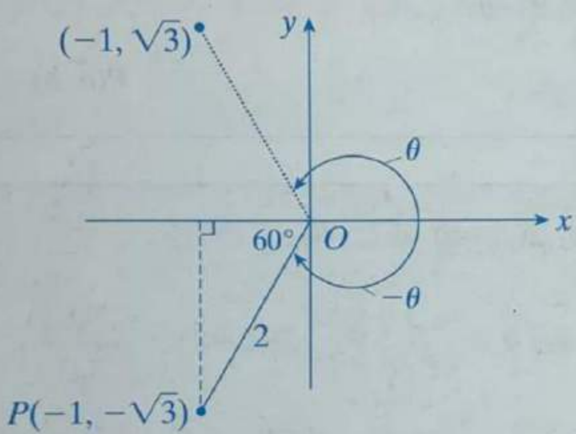


$\sin\theta = \frac{\sqrt{2}}{2}$ $\quad\quad$ $\sin(-\theta) = -\frac{\sqrt{2}}{2}$

$\cos\theta = \frac{\sqrt{2}}{2}$ $\quad\quad$ $\cos(-\theta) = \frac{\sqrt{2}}{2}$

$\tan\theta = 1$ $\quad\quad$ $\tan(-\theta) = -1$

3. Given $\theta = 120°$, find the sin, cos and tan of $(-\theta)$.

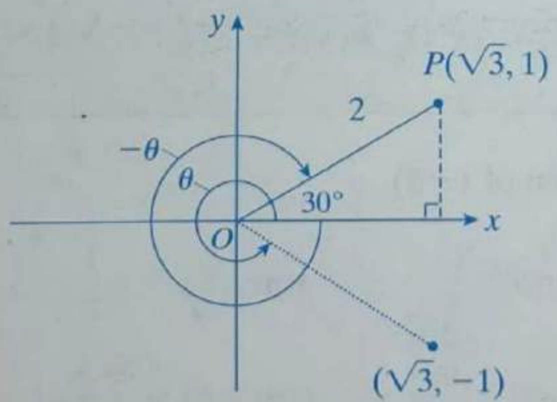


$\sin\theta = \frac{\sqrt{3}}{2}$ $\quad\quad$ $\sin(-\theta) = -\frac{\sqrt{3}}{2}$

$\cos\theta = -\frac{1}{2}$ $\quad\quad$ $\cos(-\theta) = -\frac{1}{2}$

$\tan\theta = -\sqrt{3}$ $\quad\quad$ $\tan(-\theta) = \sqrt{3}$

4. Given $\theta = 330°$, find the sin, cos and tan of $(-\theta)$.



$\sin\theta = -\frac{1}{2}$ $\quad\quad$ $\sin(-\theta) = \frac{1}{2}$

$\cos\theta = \frac{\sqrt{3}}{2}$ $\quad\quad$ $\cos(-\theta) = \frac{\sqrt{3}}{2}$

$\tan\theta = -\frac{\sqrt{3}}{3}$ $\quad\quad$ $\tan(-\theta) = \frac{\sqrt{3}}{3}$

Formulas

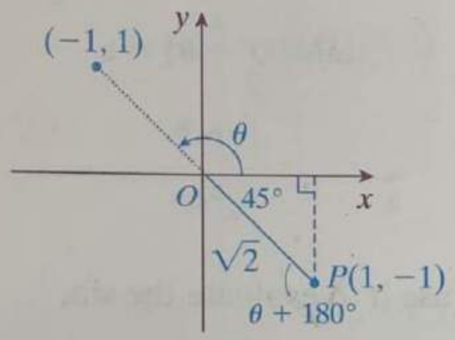
$$\boxed{\begin{aligned}\sin(-\theta) &= -\sin\theta\\ \cos(-\theta) &= \cos\theta\\ \tan(-\theta) &= -\tan\theta\end{aligned}}$$

Trigonometric Functions 3

Time   :   to   :   Date   Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. Given $\theta = 135°$, draw the diagram and use it to evaluate the sin, cos and tan of $(\theta + 180°)$.

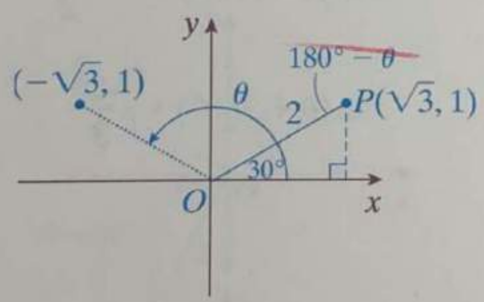


$$\sin(\theta + 180°) = -\frac{\sqrt{2}}{2}$$

$$\cos(\theta + 180°) = \frac{\sqrt{2}}{2}$$

$$\tan(\theta + 180°) = -1$$

2. Given $\theta = 150°$, draw the diagram and use it to evaluate the sin, cos and tan of $(180° - \theta)$.

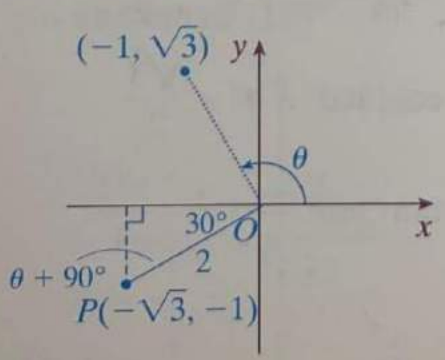


$$\sin(180° - \theta) = \frac{1}{2}$$

$$\cos(180° - \theta) = \frac{\sqrt{3}}{2}$$

$$\tan(180° - \theta) = \frac{\sqrt{3}}{3}$$

3. Given $\theta = 120°$, draw the diagram and use it to evaluate the sin, cos and tan of $(\theta + 90°)$.



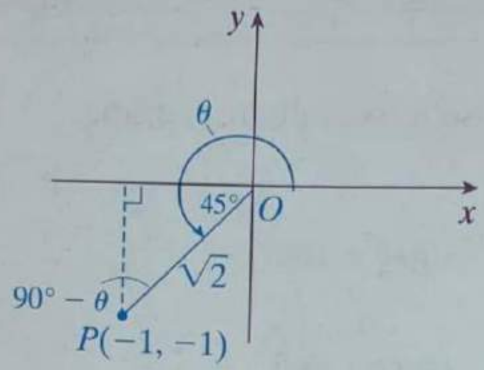
$$\sin(\theta + 90°) = -\frac{1}{2}$$

$$\cos(\theta + 90°) = -\frac{\sqrt{3}}{2}$$

$$\tan(\theta + 90°) = \frac{\sqrt{3}}{3}$$

M 27 b

4. Given $\theta = 225°$, draw the diagram and use it to evaluate the sin, cos and tan of $(90° - \theta)$.

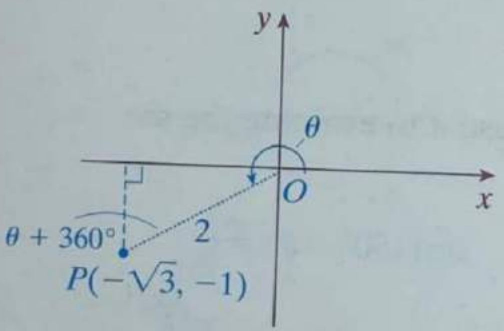


$$\sin(90° - \theta) = -\frac{\sqrt{2}}{2}$$

$$\cos(90° - \theta) = -\frac{\sqrt{2}}{2}$$

$$\tan(90° - \theta) = 1$$

5. Given $\theta = 210°$, draw the diagram and use it to evaluate the sin, cos and tan of $(\theta + 360°)$.

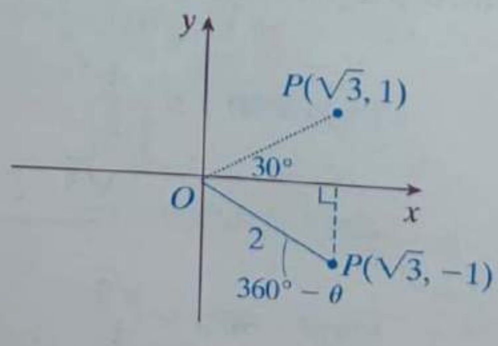


$$\sin(\theta + 360°) = -\frac{1}{2}$$

$$\cos(\theta + 360°) = -\frac{\sqrt{3}}{2}$$

$$\tan(\theta + 360°) = \frac{\sqrt{3}}{3}$$

6. Given $\theta = 30°$, draw the diagram and use it to evaluate the sin, cos and tan of $(360° - \theta)$.



$$\sin(360° - \theta) = -\frac{1}{2}$$

$$\cos(360° - \theta) = \frac{\sqrt{3}}{2}$$

$$\tan(360° - \theta) = -\frac{\sqrt{3}}{3}$$

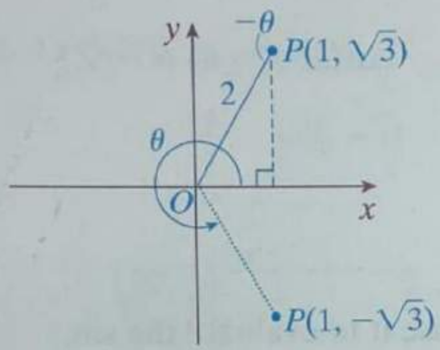
KUMON

M 28 a Trigonometric Functions 3

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. Given $\theta = 300^\circ$, draw the diagram and use it to evaluate the sin, cos and tan of $(-\theta)$.

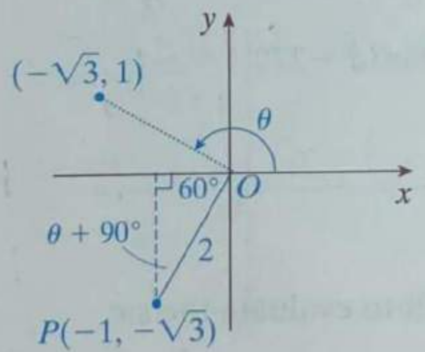


$\sin(-\theta) = \dfrac{\sqrt{3}}{2}$

$\cos(-\theta) = \dfrac{1}{2}$

$\tan(-\theta) = \sqrt{3}$

2. Given $\theta = 150^\circ$, draw the diagram and use it to evaluate the sin, cos and tan of $(\theta + 90^\circ)$.

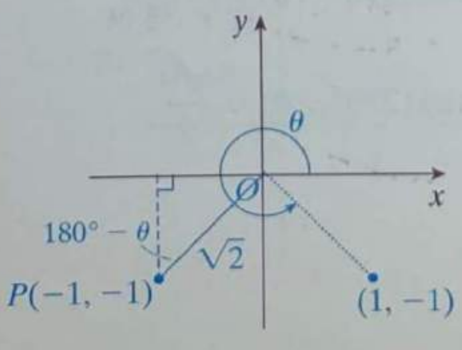


$\sin(\theta + 90^\circ) = -\dfrac{\sqrt{3}}{2}$

$\cos(\theta + 90^\circ) = -\dfrac{1}{2}$

$\tan(\theta + 90^\circ) = \sqrt{3}$

3. Given $\theta = 315^\circ$, draw the diagram and use it to evaluate the sin, cos and tan of $(180^\circ - \theta)$.

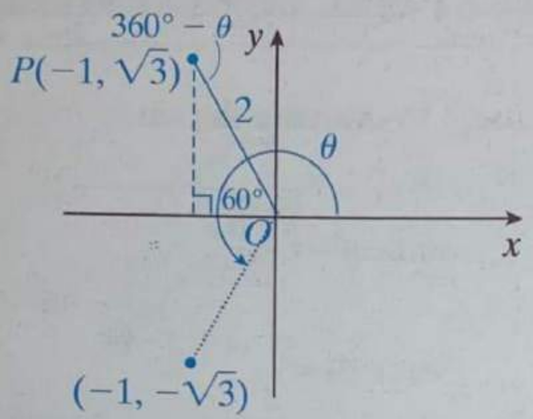


$\sin(180^\circ - \theta) = -\dfrac{\sqrt{2}}{2}$

$\cos(180^\circ - \theta) = -\dfrac{\sqrt{2}}{2}$

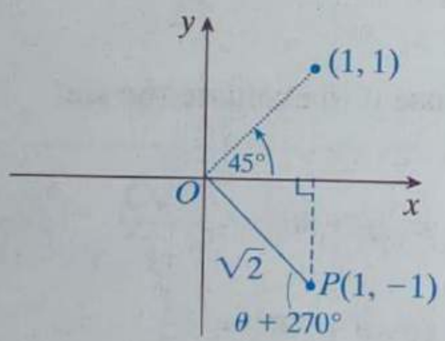
$\tan(180^\circ - \theta) = 1$

4. Given $\theta = 240°$, draw the diagram and use it to evaluate the sin, cos and tan of $(360° - \theta)$.



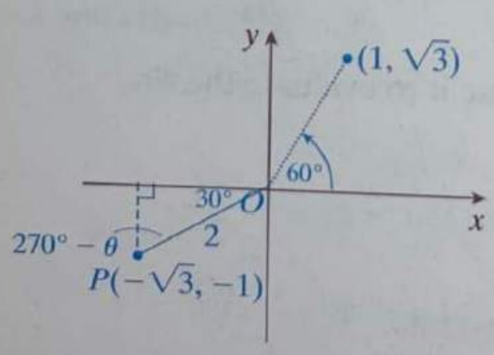
$$\sin(360° - \theta) = \frac{\sqrt{3}}{2}$$
$$\cos(360° - \theta) = -\frac{1}{2}$$
$$\tan(360° - \theta) = -\sqrt{3}$$

5. Given $\theta = 45°$, draw the diagram and use it to evaluate the sin, cos and tan of $(\theta + 270°)$.



$$\sin(\theta + 270°) = -\frac{\sqrt{2}}{2}$$
$$\cos(\theta + 270°) = \frac{\sqrt{2}}{2}$$
$$\tan(\theta + 270°) = -1$$

6. Given $\theta = 60°$, draw the diagram and use it to evaluate the sin, cos and tan of $(270° - \theta)$.



$$\sin(270° - \theta) = -\frac{1}{2}$$
$$\cos(270° - \theta) = -\frac{\sqrt{3}}{2}$$
$$\tan(270° - \theta) = \frac{\sqrt{3}}{3}$$

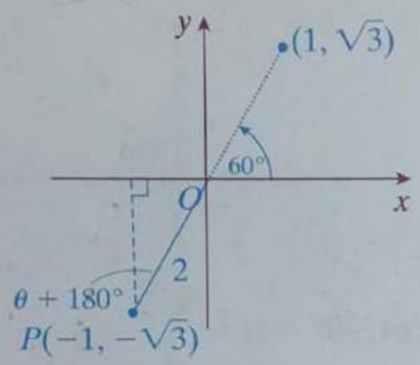
M 29 a Trigonometric Functions 3

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

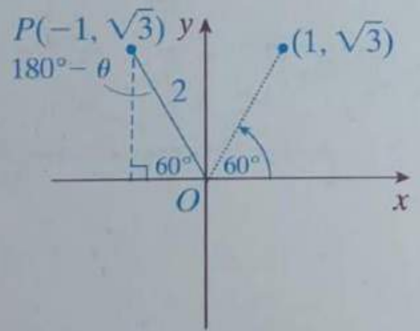
In each exercise, given that $\theta = 60°$, draw the diagram and use it to evaluate the given trigonometric expression.

(1) $\sin(\theta + 180°)$



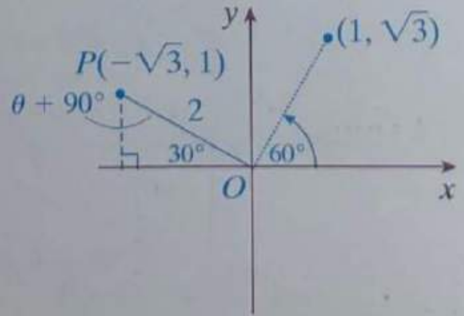
$\sin(\theta + 180°) = -\frac{\sqrt{3}}{2}$

(2) $\sin(180° - \theta)$



$\sin(180° - \theta) = \frac{\sqrt{3}}{2}$

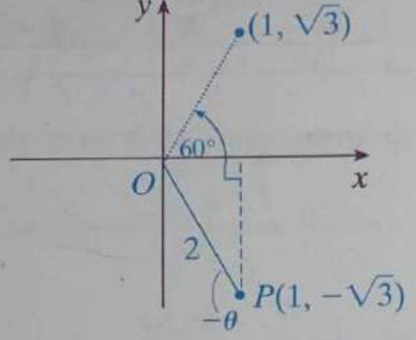
(3) $\sin(\theta + 90°)$



$\sin(\theta + 90°) = \frac{1}{2}$

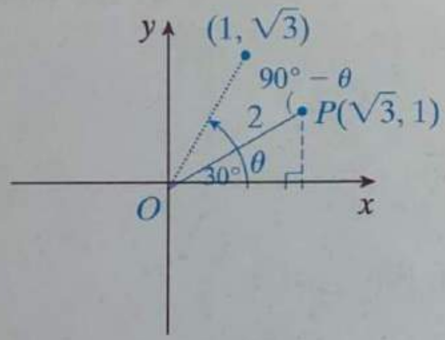
M 29 b

(4) $\cos(-\theta)$



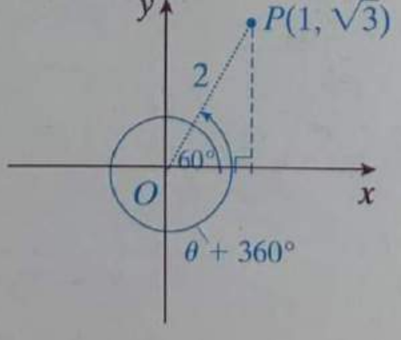
$\cos(-\theta) = \dfrac{1}{2}$

(5) $\cos(90° - \theta)$



$\cos(90° - \theta) = \dfrac{\sqrt{3}}{2}$

(6) $\cos(\theta + 360°)$



$\cos(\theta + 360°) = \dfrac{1}{2}$

M 30 a

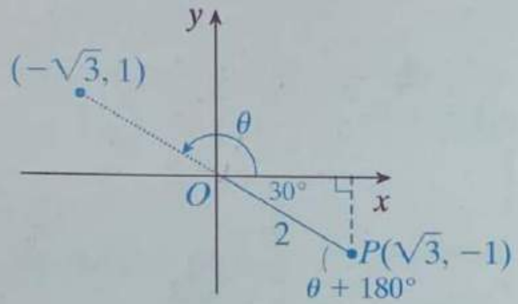
Trigonometric Functions 3

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

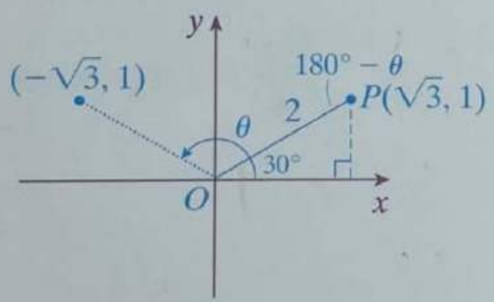
In each exercise, given that $\theta = 150°$, draw the diagram and use it to evaluate the given trigonometric expression.

(1) $\cos(\theta + 180°)$



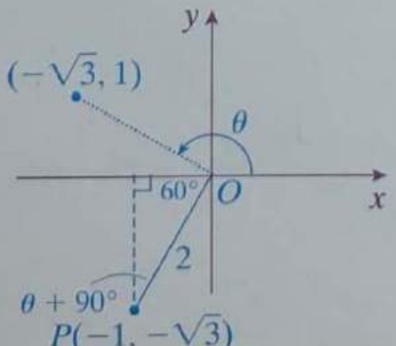
$$\cos(\theta + 180°) = \frac{\sqrt{3}}{2}$$

(2) $\tan(180° - \theta)$



$$\tan(180° - \theta) = \frac{\sqrt{3}}{3}$$

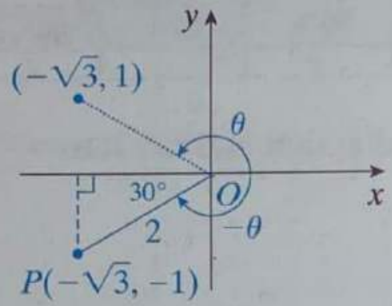
(3) $\cos(\theta + 90°)$



$$\cos(\theta + 90°) = -\frac{1}{2}$$

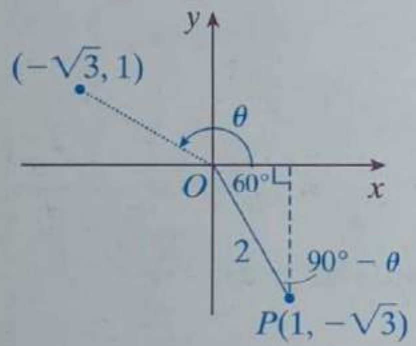
M 30 b

(4) $\tan(-\theta)$



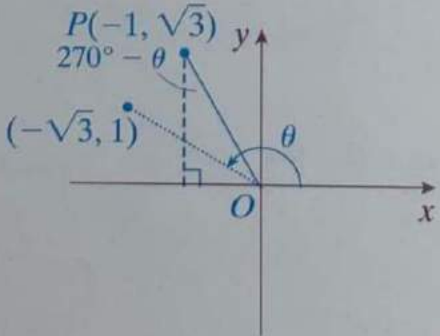
$$\tan(-\theta) = \frac{\sqrt{3}}{3}$$

(5) $\sin(90° - \theta)$



$$\sin(90° - \theta) = -\frac{\sqrt{3}}{2}$$

(6) $\sin(270° - \theta)$



$$\sin(270° - \theta) = \frac{\sqrt{3}}{2}$$

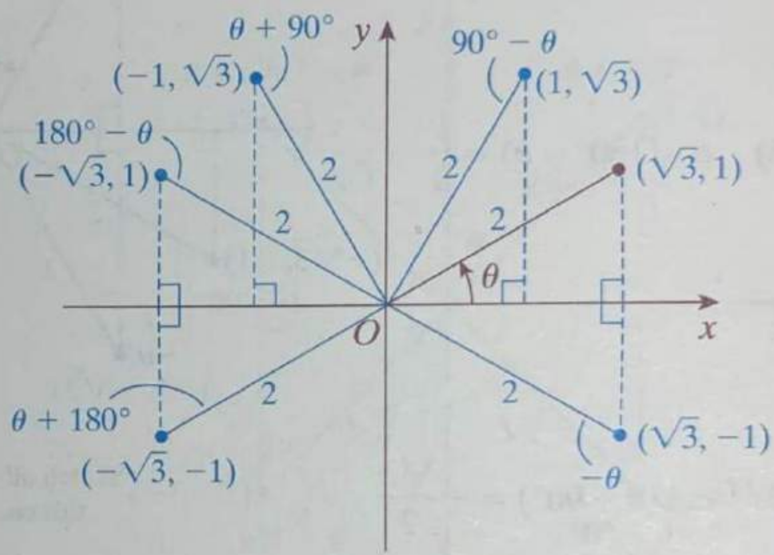
M 31 a Trigonometric Functions 4

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	1	2	3	4~

1. Given that $\theta = 30°$, draw the diagram and use it to evaluate each of the following trigonometric expressions.

(1) $\sin(\theta + 180°) = -\frac{1}{2}$



Sketch all the diagrams on this coordinate grid.

(2) $\sin(180° - \theta) = \frac{1}{2}$

(3) $\sin(\theta + 90°) = \frac{\sqrt{3}}{2}$

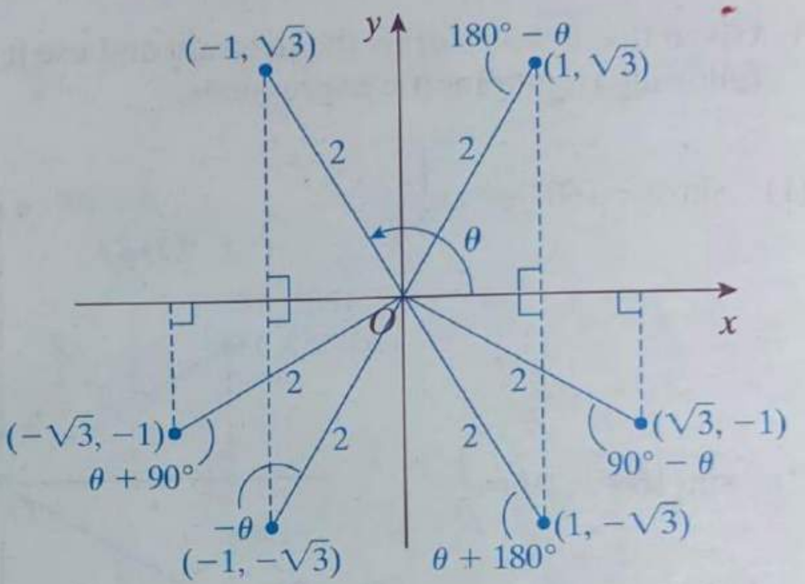
(4) $\sin(90° - \theta) = \frac{\sqrt{3}}{2}$

(5) $\sin(-\theta) = -\frac{1}{2}$

M 31 b

2. Given that $\theta = 120°$, draw the diagram and use it to evaluate each of the following trigonometric expressions.

(1) $\cos(\theta + 180°) = \frac{1}{2}$



Sketch all the diagrams on this coordinate grid.

(2) $\cos(180° - \theta) = \frac{1}{2}$

(3) $\cos(\theta + 90°) = -\frac{\sqrt{3}}{2}$

(4) $\cos(90° - \theta) = \frac{\sqrt{3}}{2}$

(5) $\cos(-\theta) = -\frac{1}{2}$

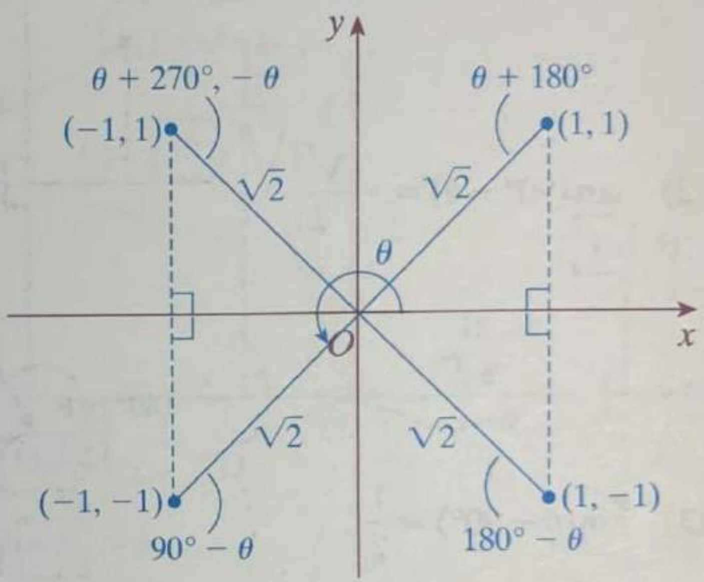
Trigonometric Functions 4

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	1	2	3	4~

1. Given that $\theta = 225°$, draw the diagram and use it to evaluate each of the following trigonometric expressions.

(1) $\cos(\theta + 180°) = \frac{\sqrt{2}}{2}$



Sketch all the diagrams on this coordinate grid.

(2) $\tan(180° - \theta) = -1$

(3) $\sin(\theta + 270°) = \frac{\sqrt{2}}{2}$

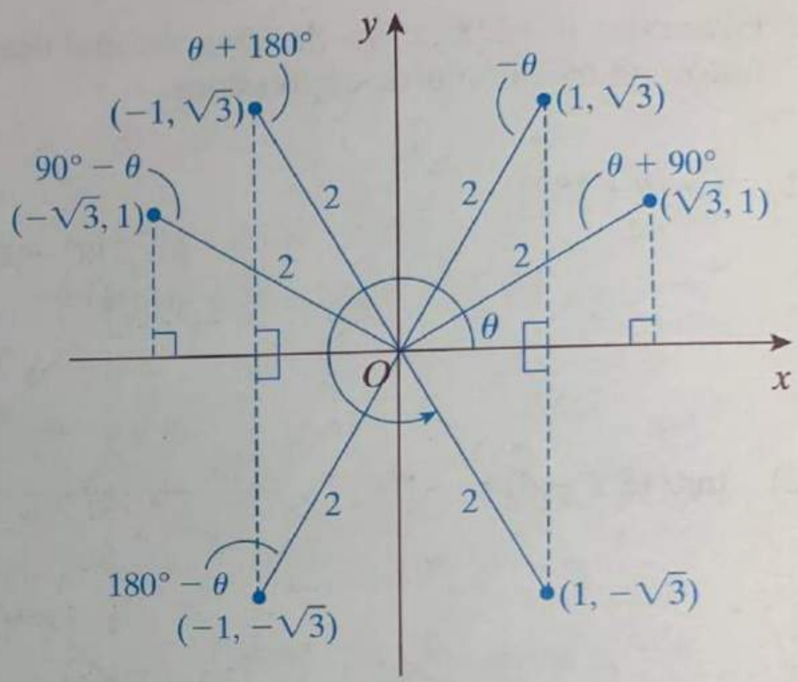
(4) $\tan(90° - \theta) = 1$

(5) $\cos(-\theta) = -\frac{\sqrt{2}}{2}$

M 32 b

2. Given that $\theta = 300°$, draw the diagram and use it to evaluate each of the following trigonometric expressions.

(1) $\sin(\theta + 180°) = \frac{\sqrt{3}}{2}$



(2) $\cos(90° - \theta) = -\frac{\sqrt{3}}{2}$

Sketch all the diagrams on this coordinate grid.

(3) $\sin(\theta + 90°) = \frac{1}{2}$

(4) $\cos(180° - \theta) = -\frac{1}{2}$

(5) $\tan(-\theta) = \sqrt{3}$

KUMON

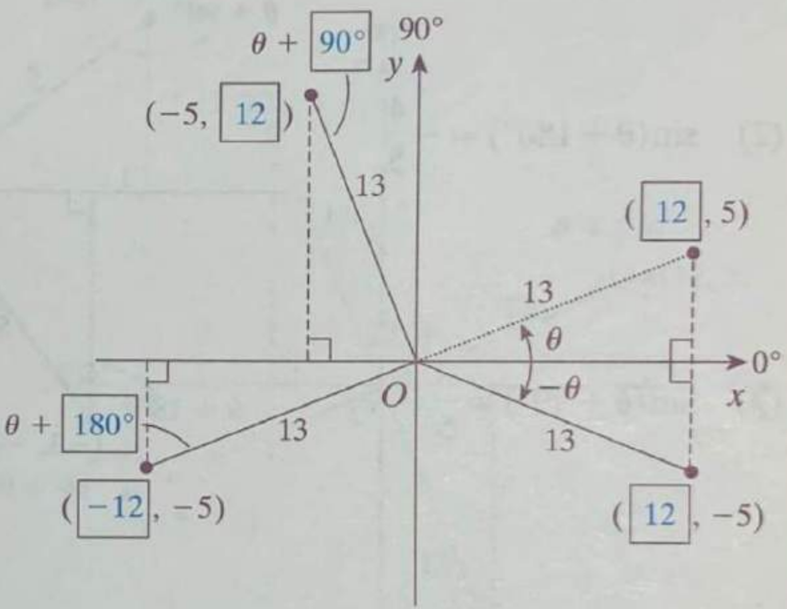
M 33 a

Trigonometric Functions 4

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	1	2	3	4~

1. Given that $0° < \theta < 90°$, and $\sin\theta = \frac{5}{13}$, complete the diagram and use it to evaluate each of the following trigonometric expressions.



(1) $\sin(-\theta) = -\frac{5}{13}$

(2) $\sin(\theta + 180°) = -\frac{5}{13}$

(3) $\sin(\theta + 90°) = \frac{12}{13}$

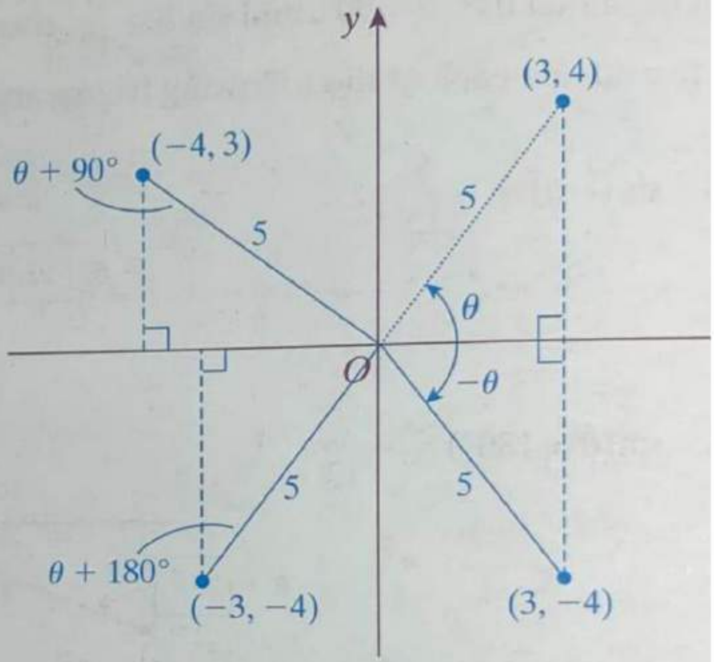
(4) $\cos(-\theta) = \frac{12}{13}$

(5) $\cos(\theta + 180°) = -\frac{12}{13}$

(6) $\cos(\theta + 90°) = -\frac{5}{13}$

M 33 b

2. Given that $0° < \theta < 90°$, and $\cos\theta = \frac{3}{5}$, draw the diagram and use it to evaluate each of the following trigonometric expressions.



(1) $\sin(-\theta) = -\frac{4}{5}$

(2) $\sin(\theta + 180°) = -\frac{4}{5}$

(3) $\sin(\theta + 90°) = \frac{3}{5}$

(4) $\cos(-\theta) = \frac{3}{5}$

(5) $\cos(\theta + 180°) = -\frac{3}{5}$

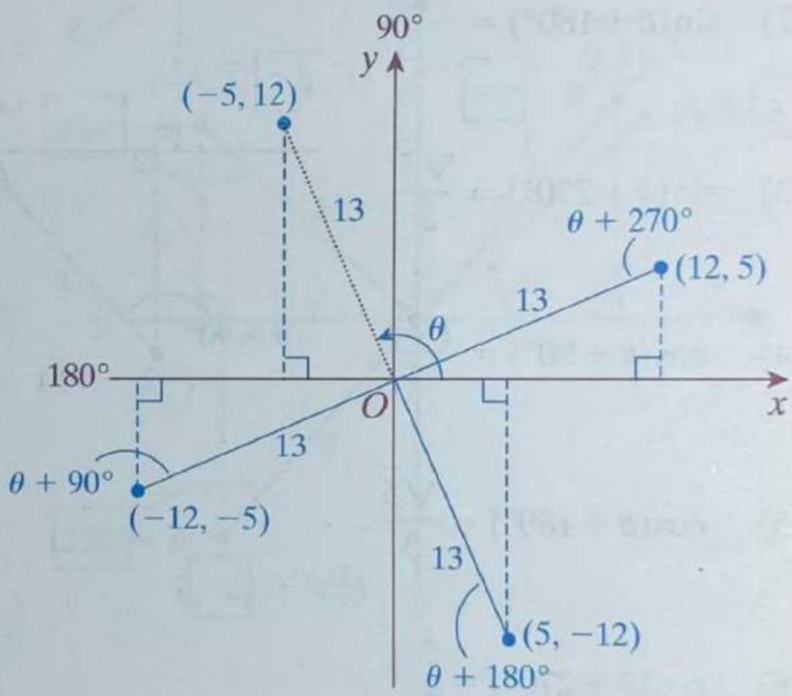
(6) $\cos(\theta + 90°) = -\frac{4}{5}$

Trigonometric Functions 4

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	1	2~3	4~5	6~

1. Given that $90° < \theta < 180°$, and $\cos\theta = -\frac{5}{13}$, draw the diagram and use it to evaluate each of the following trigonometric expressions.



(1) $\sin(\theta + 90°) = -\frac{5}{13}$

(2) $\sin(\theta + 180°) = -\frac{12}{13}$

(3) $\sin(\theta + 270°) = \frac{5}{13}$

(4) $\cos(\theta + 90°) = -\frac{12}{13}$

(5) $\cos(\theta + 180°) = \frac{5}{13}$

(6) $\cos(\theta + 270°) = \frac{12}{13}$

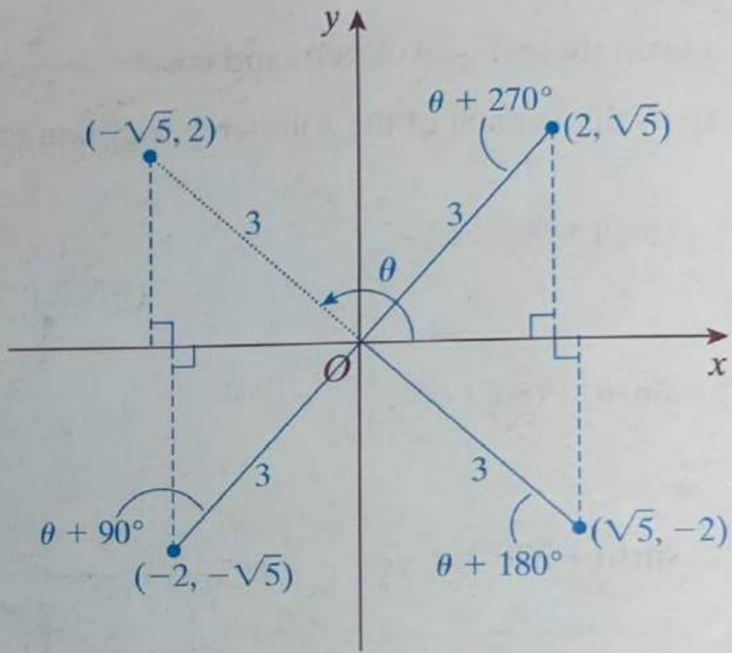
(7) $\tan(\theta + 90°) = \frac{5}{12}$

(8) $\tan(\theta + 180°) = -\frac{12}{5}$

(9) $\tan(\theta + 270°) = \frac{5}{12}$

M 34 b

2. Given that $90° < \theta < 180°$, and $\sin\theta = \frac{2}{3}$, draw the diagram and use it to evaluate each of the following trigonometric expressions.



(1) $\sin(\theta + 90°) = -\frac{\sqrt{5}}{3}$

(2) $\sin(\theta + 180°) = -\frac{2}{3}$

(3) $\sin(\theta + 270°) = \frac{\sqrt{5}}{3}$

(4) $\cos(\theta + 90°) = -\frac{2}{3}$

(5) $\cos(\theta + 180°) = \frac{\sqrt{5}}{3}$

(6) $\cos(\theta + 270°) = \frac{2}{3}$

(7) $\tan(\theta + 90°) = \frac{\sqrt{5}}{2}$

(8) $\tan(\theta + 180°) = -\frac{2\sqrt{5}}{5}$

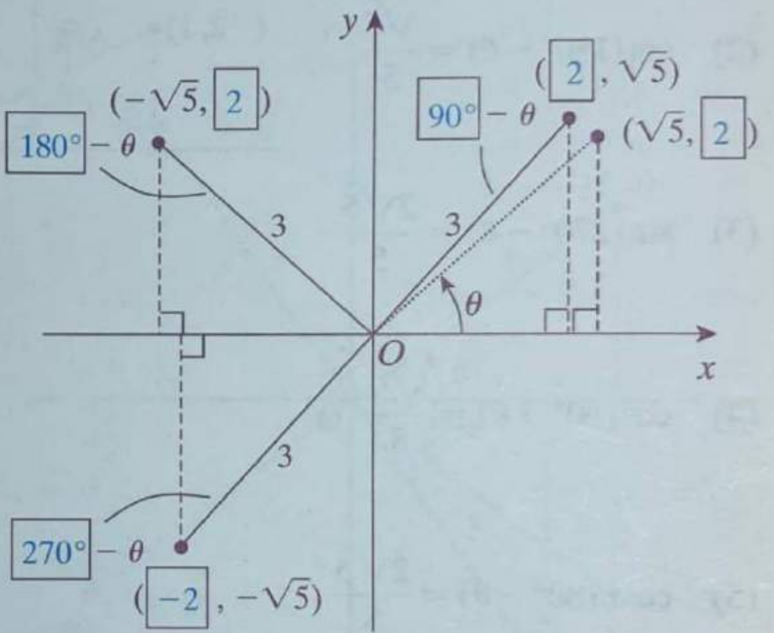
(9) $\tan(\theta + 270°) = \frac{\sqrt{5}}{2}$

Trigonometric Functions 4

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	1	2~3	4~5	6~

1. Given that $0° < \theta < 90°$, and $\cos\theta = \dfrac{\sqrt{5}}{3}$, complete the diagram and use it to evaluate each of the following trigonometric expressions.



(1) $\sin(90° - \theta) = \dfrac{\sqrt{5}}{3}$

(2) $\sin(180° - \theta) = \dfrac{2}{3}$

(3) $\sin(270° - \theta) = -\dfrac{\sqrt{5}}{3}$

(4) $\cos(90° - \theta) = \dfrac{2}{3}$

(5) $\cos(180° - \theta) = -\dfrac{\sqrt{5}}{3}$

(6) $\cos(270° - \theta) = -\dfrac{2}{3}$

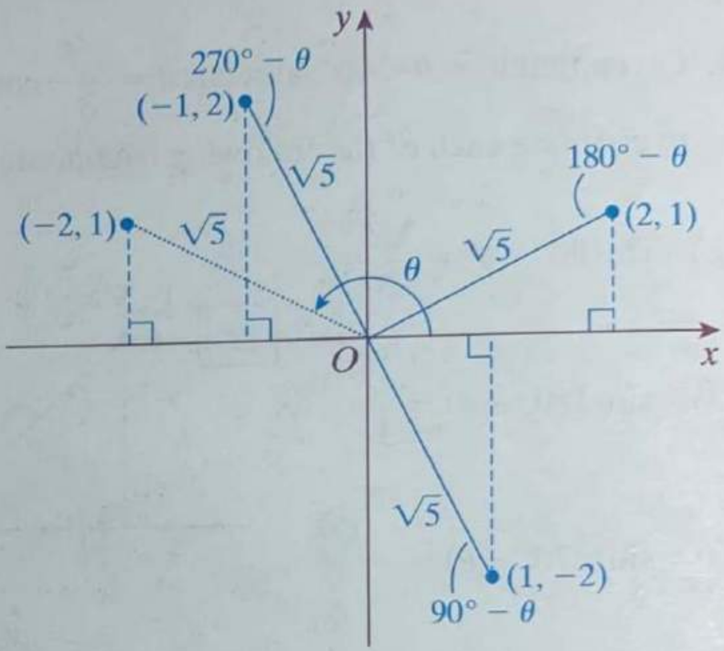
(7) $\tan(90° - \theta) = \dfrac{\sqrt{5}}{2}$

(8) $\tan(180° - \theta) = -\dfrac{2\sqrt{5}}{5}$

(9) $\tan(270° - \theta) = \dfrac{\sqrt{5}}{2}$

M 35 b

2. Given that $90° < \theta < 180°$, and $\tan\theta = -\frac{1}{2}$, draw the diagram and use it to evaluate each of the following trigonometric expressions.



(1) $\sin(90° - \theta) = -\frac{2\sqrt{5}}{5}$

(2) $\sin(180° - \theta) = \frac{\sqrt{5}}{5}$

(3) $\sin(270° - \theta) = \frac{2\sqrt{5}}{5}$

(4) $\cos(90° - \theta) = \frac{\sqrt{5}}{5}$

(5) $\cos(180° - \theta) = \frac{2\sqrt{5}}{5}$

(6) $\cos(270° - \theta) = -\frac{\sqrt{5}}{5}$

(7) $\tan(90° - \theta) = -2$

(8) $\tan(180° - \theta) = \frac{1}{2}$

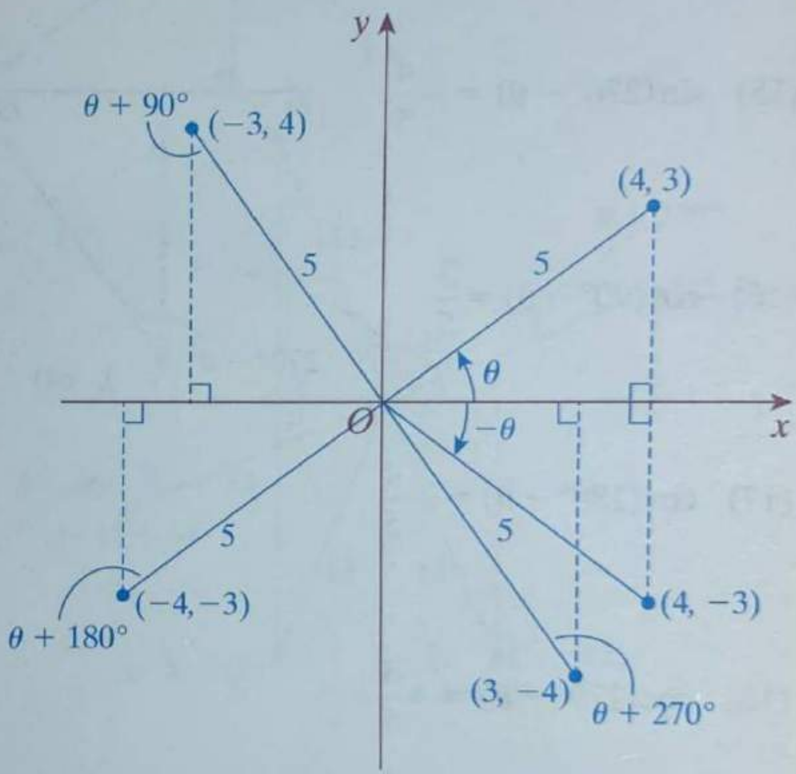
(9) $\tan(270° - \theta) = -2$

M 36 a Trigonometric Functions 4

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	1~2	3~4	5~6	7~

Given that $0° < \theta < 90°$, and $\sin\theta = \frac{3}{5}$, draw the diagram and use it to evaluate each of the following trigonometric expressions.



(1) $\sin(-\theta) = -\frac{3}{5}$

(2) $\sin(\theta + 90°) = \frac{4}{5}$

(3) $\sin(\theta + 180°) = -\frac{3}{5}$

(4) $\sin(\theta + 270°) = -\frac{4}{5}$

(5) $\cos\theta = \frac{4}{5}$

(6) $\cos(-\theta) = \frac{4}{5}$

(7) $\cos(\theta + 90°) = -\frac{3}{5}$

(8) $\cos(\theta + 180°) = -\frac{4}{5}$

(9) $\cos(\theta + 270°) = \frac{3}{5}$

(10) $\tan\theta = \frac{3}{4}$

(11) $\tan(\theta + 90°) = -\frac{4}{3}$

(12) $\tan(\theta + 180°) = \frac{3}{4}$

M 36 b

(13) $\sin(90° - \theta) = \frac{4}{5}$

(14) $\sin(180° - \theta) = \frac{3}{5}$

(15) $\sin(270° - \theta) = -\frac{4}{5}$

(16) $\cos(90° - \theta) = \frac{3}{5}$

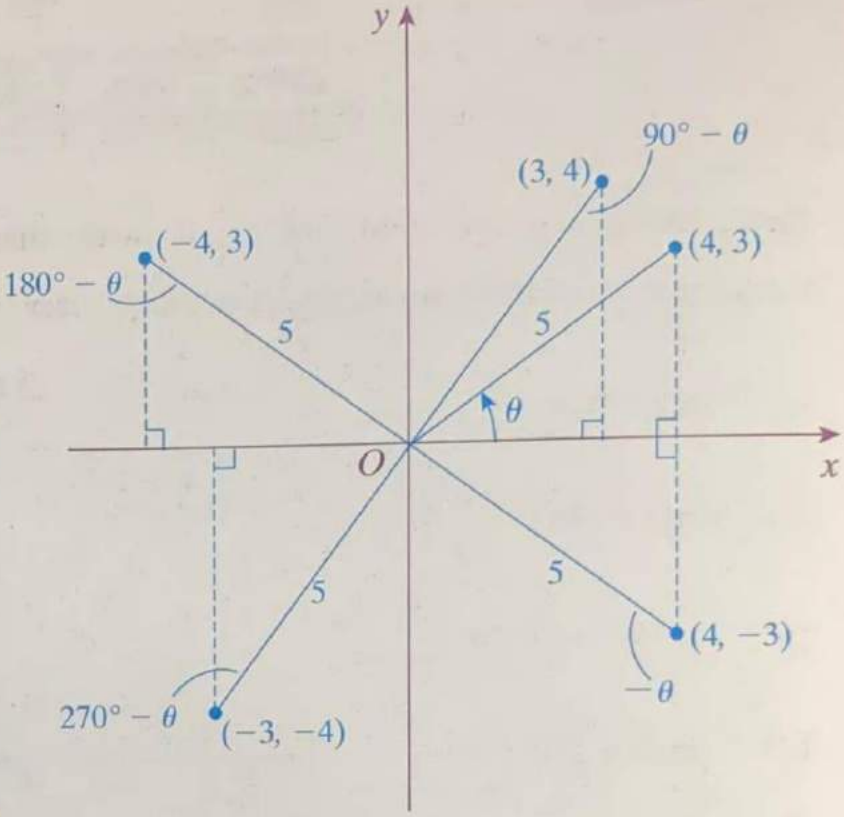
(17) $\cos(180° - \theta) = -\frac{4}{5}$

(18) $\cos(270° - \theta) = -\frac{3}{5}$

(19) $\tan(-\theta) = -\frac{3}{4}$

(20) $\tan(90° - \theta) = \frac{4}{3}$

(21) $\tan(180° - \theta) = -\frac{3}{4}$

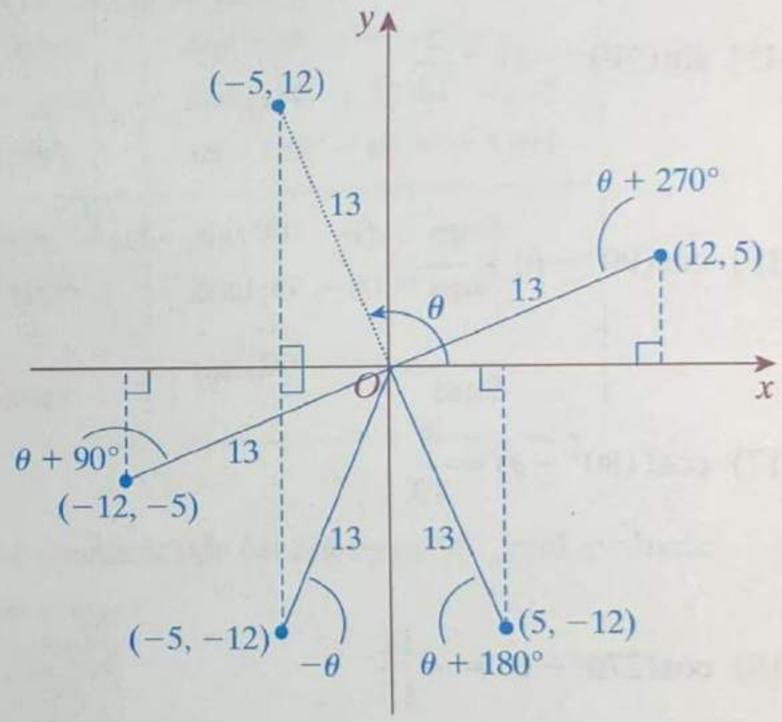


M 37 a Trigonometric Functions 4

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	1~2	3~4	5~6	7~

Given that $90° < \theta < 180°$, and $\tan\theta = -\frac{12}{5}$, draw the diagram and use it to evaluate each of the following trigonometric expressions.



(1) $\sin\theta = \frac{12}{13}$

(2) $\sin(-\theta) = -\frac{12}{13}$

(3) $\sin(\theta + 90°) = -\frac{5}{13}$

(4) $\sin(\theta + 180°) = -\frac{12}{13}$

(5) $\sin(\theta + 270°) = \frac{5}{13}$

(6) $\cos\theta = -\frac{5}{13}$

(7) $\cos(-\theta) = -\frac{5}{13}$

(8) $\cos(\theta + 90°) = -\frac{12}{13}$

(9) $\cos(\theta + 180°) = \frac{5}{13}$

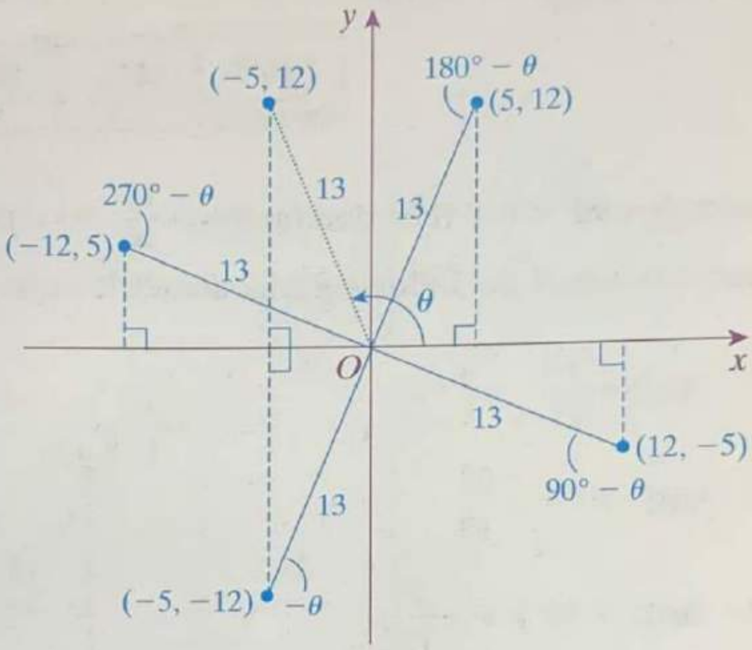
(10) $\cos(\theta + 270°) = \frac{12}{13}$

(11) $\tan(\theta + 90°) = \frac{5}{12}$

(12) $\tan(\theta + 180°) = -\frac{12}{5}$

M 37 b

(13) $\sin(90^\circ - \theta) = -\dfrac{5}{13}$



(14) $\sin(180^\circ - \theta) = \dfrac{12}{13}$

(15) $\sin(270^\circ - \theta) = \dfrac{5}{13}$

(16) $\cos(90^\circ - \theta) = \dfrac{12}{13}$

(17) $\cos(180^\circ - \theta) = \dfrac{5}{13}$

(18) $\cos(270^\circ - \theta) = -\dfrac{12}{13}$

(19) $\tan(-\theta) = \dfrac{12}{5}$

(20) $\tan(90^\circ - \theta) = -\dfrac{5}{12}$

(21) $\tan(180^\circ - \theta) = \dfrac{12}{5}$

M 38 a Trigonometric Functions 4

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	1	2~3	4~5	6~

Review Formulas	$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = \cos\theta$ $\tan(-\theta) = -\tan\theta$
$\sin(\theta+180°) = -\sin\theta$ $\cos(\theta+180°) = -\cos\theta$ $\tan(\theta+180°) = \tan\theta$	$\sin(180°-\theta) = \sin\theta$ $\cos(180°-\theta) = -\cos\theta$ $\tan(180°-\theta) = -\tan\theta$
$\sin(\theta+90°) = \cos\theta$ $\cos(\theta+90°) = -\sin\theta$ $\tan(\theta+90°) = -\frac{1}{\tan\theta}$	$\sin(90°-\theta) = \cos\theta$ $\cos(90°-\theta) = \sin\theta$ $\tan(90°-\theta) = \frac{1}{\tan\theta}$

1. Express the following as trigonometric functions of 30°, and evaluate.

Ex. $\boxed{\sin 60° = \cos 30° = \frac{\sqrt{3}}{2}}$

(1) $\cos 60° = \sin 30° = \frac{1}{2}$

(2) $\tan 120° = -\tan 60°$

$= -\frac{1}{\tan 30°} = -\sqrt{3}$

(3) $\sin 120° = \sin 60° = \cos 30° = \frac{\sqrt{3}}{2}$

(4) $\tan 150° = -\tan 30° = -\frac{\sqrt{3}}{3}$

(5) $\cos 150° = -\cos 30° = -\frac{\sqrt{3}}{2}$

2. Rewrite each of the following as a trigonometric expression of a positive angle less than 45°.

Ex. $\sin 72° = \sin(90° - 18°) = \cos 18°$

(1) $\cos 78° = \cos(90° - 12°) = \sin 12°$

(2) $\tan 65° = \tan(90° - 25°) = \dfrac{1}{\tan 25°}$

(3) $\sin 134° = \sin(90° + 44°) = \cos 44°$

(4) $\cos 165° = \cos(180° - 15°) = -\cos 15°$

(5) $\tan 100° = \tan(90° + 10°) = -\dfrac{1}{\tan 10°}$

(6) $\sin 168° = \sin(180° - 12°) = \sin 12°$

(7) $\cos 106° = \cos(90° + 16°) = -\sin 16°$

(8) $\tan 224° = \tan(180° + 44°) = \tan 44°$

3. Evaluate the following expressions.

(1) $\sin(-390°) = -\sin 390° = -\sin 30° = \boxed{-\dfrac{1}{2}}$

(2) $\cos(-120°) = \cos 120° = -\cos 60° = -\dfrac{1}{2}$

(3) $\cos(-300°) = \cos 300° = \cos 60° = \dfrac{1}{2}$

(4) $\tan(-225°) = -\tan 225° = -\tan 45° = -1$

(5) $\tan(-315°) = -\tan 315° = -(-\tan 45°) = \tan 45° = 1$

M 39 a

Trigonometric Functions 4

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. Simplify the following expressions.

Ex.

$\tan\alpha\sin(90^\circ-\alpha)-\cos^2\alpha\cos(90^\circ-\alpha)$

$=\dfrac{\sin\alpha}{\cos\alpha}\cdot\cos\alpha-\cos^2\alpha\sin\alpha$

From: $\tan\alpha=\dfrac{\sin\alpha}{\cos\alpha}$

$\sin(90^\circ-\alpha)=\cos\alpha$

$\cos(90^\circ-\alpha)=\sin\alpha$

$=\sin\alpha-\cos^2\alpha\sin\alpha$

Factor.

$=\sin\alpha(1-\cos^2\alpha)$

From $\sin^2\alpha+\cos^2\alpha=1$,

$\sin^2\alpha=1-\cos^2\alpha$

$=\sin\alpha(\sin^2\alpha)$

$=\sin^3\alpha$

(1) $(\sin\alpha+\cos\alpha)^2+(\sin\alpha-\cos\alpha)^2$

$=1+2\sin\alpha\cos\alpha+1-2\sin\alpha\cos\alpha$

$=2$

(2) $\dfrac{1-\sin\alpha}{1+\sin\alpha}+\dfrac{1+\sin\alpha}{1-\sin\alpha}-2$

$=\dfrac{2+2\sin^2\alpha}{1-\sin^2\alpha}-2$

$=\dfrac{2+2\sin^2\alpha-2+2\sin^2\alpha}{1-\sin^2\alpha}$

$=\dfrac{4\sin^2\alpha}{\cos^2\alpha}$

$=4\tan^2\alpha$

M 39 b

2. Prove the following equalities.

(1) $1+\dfrac{1}{\tan^2 A}=\dfrac{1}{\sin^2 A}$

[Sol] $\text{LHS}=1+\dfrac{1}{\left(\dfrac{\boxed{\sin A}}{\cos A}\right)^2}=1+\dfrac{\boxed{\cos^2 A}}{\sin^2 A}$

$=\dfrac{\boxed{\sin^2 A}+\cos^2 A}{\sin^2 A}=\dfrac{\boxed{1}}{\sin^2 A}=\text{RHS}$

(2) $\dfrac{\sin(90^\circ - A)}{\sin A}+\dfrac{\sin A}{\sin(90^\circ - A)}=\dfrac{1}{\sin A\cos A}$

[Sol] $\text{LHS}=\dfrac{\cos A}{\sin A}+\dfrac{\sin A}{\cos A}=\dfrac{\cos^2 A+\sin^2 A}{\sin A\cos A}$

$=\dfrac{1}{\sin A\cos A}=\text{RHS}$

(3) $\dfrac{1}{\tan^2(90^\circ - A)}+1=\dfrac{1}{\cos^2 A}$

[Sol] $\text{LHS}=\dfrac{\cos^2(90^\circ - A)}{\sin^2(90^\circ - A)}+1$

$=\dfrac{\sin^2 A}{\cos^2 A}+1$

$=\dfrac{\sin^2 A+\cos^2 A}{\cos^2 A}$

$=\dfrac{1}{\cos^2 A}=\text{RHS}$

M 40 a Trigonometric Functions 4

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Prove the following equalities.

(1) $\tan^2 A - \tan^2 A \sin^2 A = \sin^2 A$

[Sol] LHS $= \tan^2 A(1-\sin^2 A)$

$= \dfrac{\sin^2 A}{\cos^2 A} \cdot \cos^2 A$

$= \sin^2 A = \text{RHS}$

(2) $\sin^4 A - \cos^4 A = 2\sin^2 A - 1$

[Sol] LHS $= (\sin^2 A + \cos^2 A)(\sin^2 A - \cos^2 A)$

$= \sin^2 A - \cos^2 A$

$= \sin^2 A - (1 - \sin^2 A)$

$= 2\sin^2 A - 1 = \text{RHS}$

(3) $\dfrac{1+\sin A}{\cos A} + \dfrac{\cos A}{1+\sin A} = \dfrac{2}{\cos A}$

[Sol] LHS $= \dfrac{(1+\sin A)^2 + \cos^2 A}{\cos A(1+\sin A)}$

$= \dfrac{1 + 2\sin A + \sin^2 A + \cos^2 A}{\cos A(1+\sin A)}$

$= \dfrac{2(1+\sin A)}{\cos A(1+\sin A)} = \dfrac{2}{\cos A} = \text{RHS}$

M 40 b

(4) $\dfrac{\sin A}{1+\cos A}+\dfrac{\sin A}{1-\cos A}=\dfrac{2}{\sin A}$

[Sol] $\text{LHS}=\dfrac{\sin A(1-\cos A)+\sin A(1+\cos A)}{(1+\cos A)(1-\cos A)}$

$$=\frac{2\sin A}{1-\cos^2 A}$$

$$=\frac{2\sin A}{\sin^2 A}$$

$$=\frac{2}{\sin A}=\text{RHS}$$

(5) $\dfrac{1}{1+\sin A}+\dfrac{1}{1-\sin A}=\dfrac{2}{\cos^2 A}$

[Sol] $\text{LHS}=\dfrac{(1-\sin A)+(1+\sin A)}{(1+\sin A)(1-\sin A)}$

$$=\frac{2}{1-\sin^2 A}$$

$$=\frac{2}{\cos^2 A}=\text{RHS}$$

(6) $\dfrac{1-\tan^2 A}{1+\tan^2 A}=\cos^2 A-\sin^2 A$

[Sol] $\text{LHS}=\dfrac{1-\dfrac{\sin^2 A}{\cos^2 A}}{1+\dfrac{\sin^2 A}{\cos^2 A}}$

$$=\frac{\cos^2 A-\sin^2 A}{\cos^2 A+\sin^2 A}$$

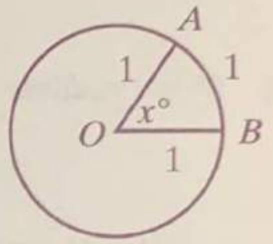
$$=\cos^2 A-\sin^2 A=\text{RHS}$$

M 41 a Trigonometric Functions 5

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	1	2~3	4~5	6~

Given a circle with radius 1 and center at point O, where OA and OB are two radii such that arc AB has length 1:



$$m\angle AOB = 1 \text{ radian}$$

If $\angle AOB$ measures $x°$, then the ratio of $x°$ to $360°$ is equal to the ratio of arc AB to the circumference, 2π.

Since arc $AB = 1$, $\dfrac{x°}{360°} = \dfrac{\boxed{1}}{2\pi}$, and $x° = \boxed{\dfrac{180°}{\pi}}$. (where $\pi \approx 3.14$)

Therefore,

Formulas

$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees}$$

Multiply both sides by $\frac{\pi}{180}$.

$$1 \text{ degree} = \boxed{\frac{\pi}{180}} \text{ radians}$$

Answers: 1, $\frac{180°}{\pi}$, $\frac{\pi}{180}$

Convert the following angles from degrees to radians.

Ex. $30° = 30° \cdot \dfrac{\pi}{180°} = \dfrac{\pi}{6}$ *radians*

(1) $90° = \dfrac{\pi}{2}$ *radians*

(2) $40° = \dfrac{2}{9}\pi$ *radians*

(3) $-10° = -\dfrac{\pi}{18}$ *radians*

Review

A circle measures 360 degrees.

The circumference of a circle is $C = 2\pi r$. When $r = 1$, $C = 2\pi$.

M 41 b

(4) $36^\circ = \frac{\pi}{5}$ *radians*

(11) $150^\circ = \frac{5}{6}\pi$ *radians*

(5) $0^\circ = 0$ *radians*

(12) $45^\circ = \frac{\pi}{4}$ *radians*

(6) $-130^\circ = -\frac{13}{18}\pi$ *radians*

(13) $60^\circ = \frac{\pi}{3}$ *radians*

(7) $54^\circ = \frac{3}{10}\pi$ *radians*

(14) $-120^\circ = -\frac{2}{3}\pi$ *radians*

(8) $100^\circ = \frac{5}{9}\pi$ *radians*

(15) $270^\circ = \frac{3}{2}\pi$ *radians*

(9) $-30^\circ = -\frac{\pi}{6}$ *radians*

(16) $360^\circ = 2\pi$ *radians*

(10) $260^\circ = \frac{13}{9}\pi$ *radians*

(17) $-72^\circ = -\frac{2}{5}\pi$ *radians*

KUMON

M 42 a Trigonometric Functions 5

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	1~3	4~6	7~9	10~

1. Convert the following angles from radians to degrees.

Ex. $\frac{\pi}{3}\ radians = \frac{\pi}{3}\cdot\frac{180°}{\pi} = 60°$

(1) $\pi\ radians = 180°$

(2) $\frac{\pi}{6}\ radians = 30°$

(3) $-\frac{\pi}{3}\ radians = -60°$

(4) $-2\pi\ radians = -360°$

(5) $\frac{5}{4}\pi\ radians = 225°$

(6) $-\frac{\pi}{4}\ radians = -45°$

(7) $\frac{\pi}{9}\ radians = 20°$

(8) $\frac{3}{2}\pi\ radians = 270°$

(9) $-\frac{\pi}{2}\ radians = -90°$

(10) $-\frac{5}{3}\pi\ radians = -300°$

(11) $\frac{7}{6}\pi\ radians = 210°$

(12) $-\frac{\pi}{5}\ radians = -36°$

(13) $0\ radians = 0°$

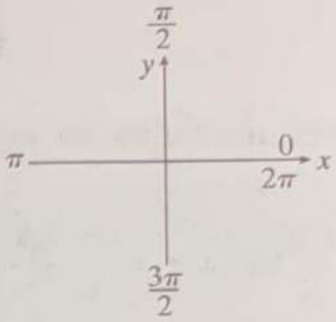
(14) $\frac{3}{4}\pi\ radians = 135°$

(15) $3\pi\ radians = 540°$

(16) $\frac{2}{3}\pi\ radians = 120°$

M 42 b

2. Simplify each of the following expressions.



[Reference]

Ex. $\boxed{\frac{\pi}{2}+\frac{\pi}{6}=\frac{4\pi}{6}=\frac{2\pi}{3}}$

(1) $\frac{\pi}{2}+\frac{\pi}{4}=\frac{3\pi}{4}$

(2) $\frac{\pi}{2}+\frac{\pi}{3}=\frac{5\pi}{6}$

(3) $\pi+\frac{\pi}{6}=\frac{7\pi}{6}$

(4) $\pi+\frac{\pi}{4}=\frac{5\pi}{4}$

(5) $\pi+\frac{\pi}{3}=\frac{4\pi}{3}$

(6) $\frac{3\pi}{2}+\frac{\pi}{6}=\frac{10\pi}{6}=\frac{5\pi}{3}$

(7) $\frac{3\pi}{2}+\frac{\pi}{4}=\frac{7\pi}{4}$

(8) $\frac{3\pi}{2}+\frac{\pi}{3}=\frac{11\pi}{6}$

(9) $\pi-\frac{\pi}{4}=\frac{3\pi}{4}$

(10) $\pi-\frac{\pi}{6}=\frac{5\pi}{6}$

(11) $\frac{3\pi}{2}-\frac{\pi}{3}=\frac{7\pi}{6}$

(12) $\frac{3\pi}{2}-\frac{\pi}{4}=\frac{5\pi}{4}$

(13) $\frac{3\pi}{2}-\frac{\pi}{6}=\frac{8\pi}{6}=\frac{4\pi}{3}$

(14) $2\pi-\frac{\pi}{3}=\frac{5\pi}{3}$

(15) $2\pi-\frac{\pi}{4}=\frac{7\pi}{4}$

(16) $2\pi-\frac{\pi}{6}=\frac{11\pi}{6}$

M 43 a   Trigonometric Functions 5

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	1	2	3	4~

1. Simplify each expression. Then, use the radian measures calculated in (1)-(6) to fill-in the blank boxes on the graph below.

Ex. $\frac{\pi}{2}+\frac{\pi}{6}=\boxed{\frac{2\pi}{3}}$

(1) $\frac{\pi}{2}+\frac{\pi}{4}=\frac{3\pi}{4}$

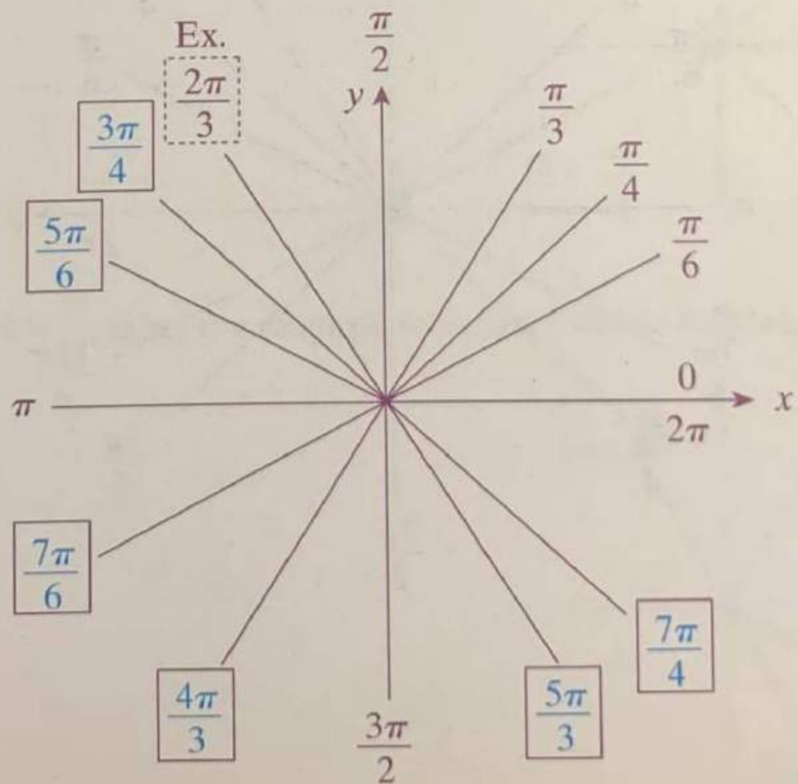
(2) $\pi+\frac{\pi}{6}=\frac{7\pi}{6}$

(3) $\frac{3\pi}{2}+\frac{\pi}{4}=\frac{7\pi}{4}$

(4) $\pi-\frac{\pi}{6}=\frac{5\pi}{6}$

(5) $\frac{3\pi}{2}-\frac{\pi}{6}=\frac{4\pi}{3}$

(6) $2\pi-\frac{\pi}{3}=\frac{5\pi}{3}$



2. Simplify each expression. Then, draw and label the diagram of each radian measure calculated in (1)-(6) on the graph below.

(1) $\frac{\pi}{2}+\frac{\pi}{3}=\frac{5\pi}{6}$

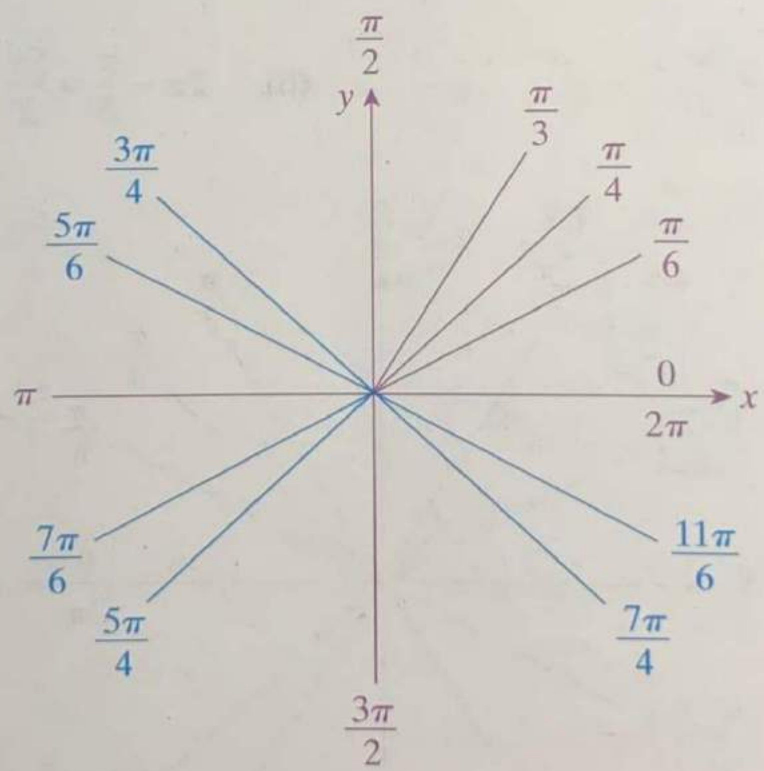
(4) $\pi-\frac{\pi}{4}=\frac{3\pi}{4}$

(2) $\pi+\frac{\pi}{4}=\frac{5\pi}{4}$

(5) $\frac{3\pi}{2}-\frac{\pi}{3}=\frac{7\pi}{6}$

(3) $\frac{3\pi}{2}+\frac{\pi}{3}=\frac{11\pi}{6}$

(6) $2\pi-\frac{\pi}{4}=\frac{7\pi}{4}$



Trigonometric Functions 5

Time : to : Date Name

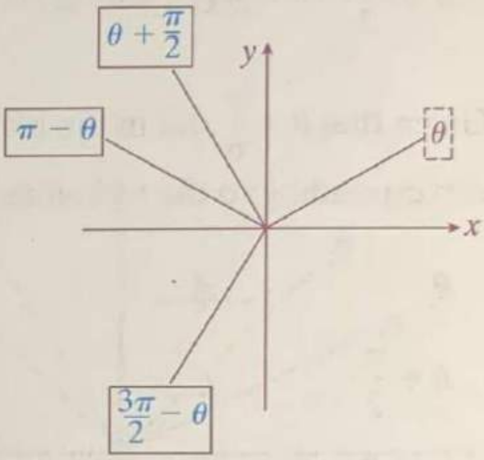
100%	90%	80%	70%	69%~
(mistakes) 0	1~2	3~5	6~7	8~

1. Given that $\theta = \frac{\pi}{6}$, label the diagram using each of the following angles.

(1) $\theta + \frac{\pi}{2}$

(2) $\pi - \theta$

(3) $\frac{3\pi}{2} - \theta$



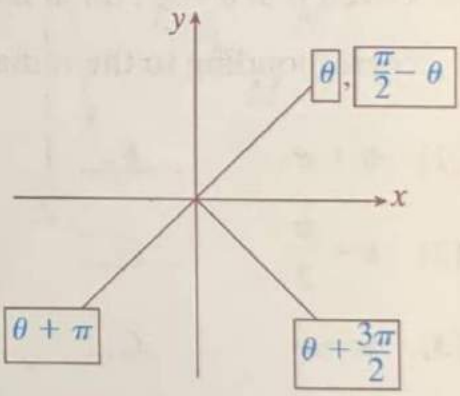
2. Given that $\theta = \frac{\pi}{4}$, label the diagram using each of the following angles.

(1) θ

(2) $\frac{\pi}{2} - \theta$

(3) $\theta + \pi$

(4) $\theta + \frac{3\pi}{2}$



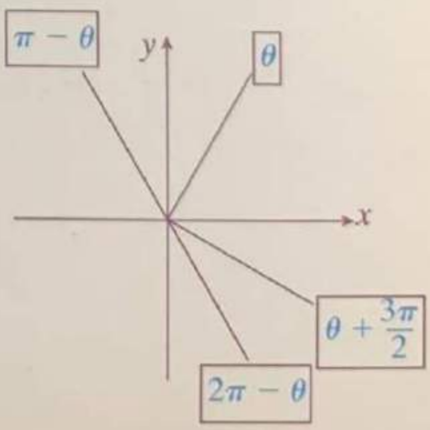
3. Given that $\theta = \frac{\pi}{3}$, label the diagram using each of the following angles.

(1) θ

(2) $\pi - \theta$

(3) $\theta + \frac{3\pi}{2}$

(4) $2\pi - \theta$



4. Given that $\theta = \frac{\pi}{4}$, fill in the blank with the appropriate letter corresponding to the radian measure on the graph below.

(1) θ B

(2) $\theta + \frac{3\pi}{2}$ G

(3) $2\pi - \theta$ G

(4) $\theta + 2\pi$ B

5. Given that $\theta = \frac{\pi}{6}$, fill in the blank with the appropriate letter corresponding to the radian measure on the graph below.

(1) θ A

(2) $\theta + \frac{\pi}{2}$ C

(3) $\pi - \theta$ D

(4) $\frac{3\pi}{2} - \theta$ E

(5) $\theta + 2\pi$ A

6. Given that $\theta = \frac{\pi}{3}$, fill in the blank with the appropriate letter corresponding to the radian measure on the graph below.

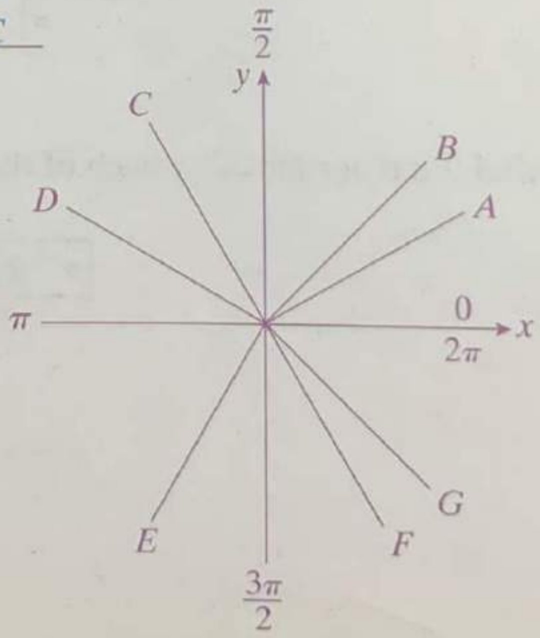
(1) $\theta + \pi$ E

(2) $\theta + \frac{\pi}{2}$ D

(3) $\pi - \theta$ C

(4) $\frac{\pi}{2} - \theta$ A

(5) $2\pi - \theta$ F



Trigonometric Functions 5

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	1~2	3~4	5~6	7~

1. For each given radian measure, fill-in the blank with the appropriate letter of the corresponding radius from the graph, as shown in the example.

Ex. $\frac{\pi}{2}$… $\boxed{D}$

(1) $\frac{\pi}{4}$… $\boxed{B}$

(2) $-\frac{\pi}{6}$… $\boxed{N}$

(3) $\frac{\pi}{3}$… $\boxed{C}$

(4) $\frac{2\pi}{3}$… $\boxed{E}$

(5) π… $\boxed{G}$

(6) $\frac{\pi}{6}$… $\boxed{A}$

(7) $\frac{3\pi}{4}$… $\boxed{F}$

(8) 2π… $\boxed{P}$

(9) $\frac{5\pi}{4}$… $\boxed{I}$

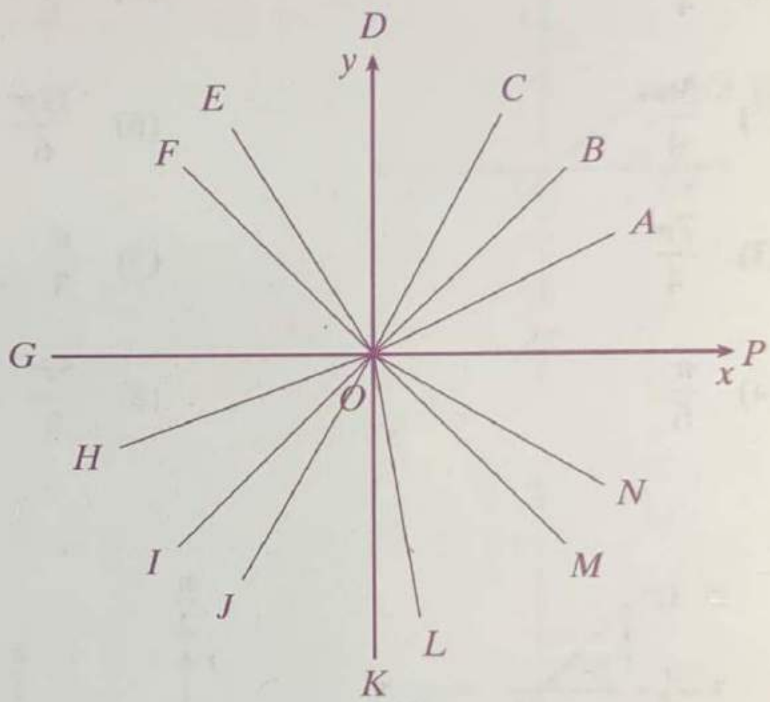
(10) $\frac{4\pi}{3}$… $\boxed{J}$

(11) $\frac{7\pi}{4}$… $\boxed{M}$

(12) $\frac{3\pi}{2}$… $\boxed{K}$

(13) $-\frac{4\pi}{9}$… $\boxed{L}$

(14) $\frac{10\pi}{9}$… $\boxed{H}$



2. Draw and label each radian measure on the graph below.

Ex. $\boxed{\dfrac{4\pi}{3}}$

(1) $\dfrac{\pi}{4}$

(2) $\dfrac{3\pi}{4}$

(3) $\dfrac{7\pi}{4}$

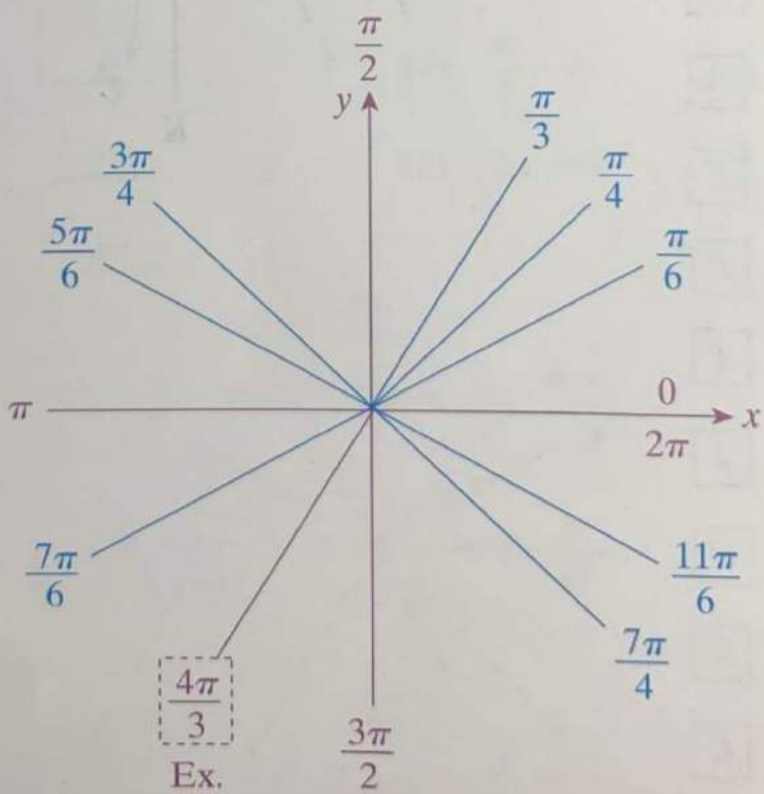
(4) $\dfrac{\pi}{6}$

(5) $\dfrac{7\pi}{6}$

(6) $\dfrac{11\pi}{6}$

(7) $\dfrac{\pi}{3}$

(8) $\dfrac{5\pi}{6}$



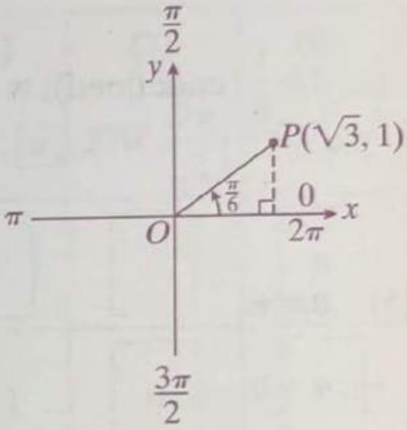
Trigonometric Functions 5

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

In each of the following exercises, draw the diagram, and use it to evaluate the trigonometric functions of the given angle.

(1) $\theta = \frac{\pi}{6}$

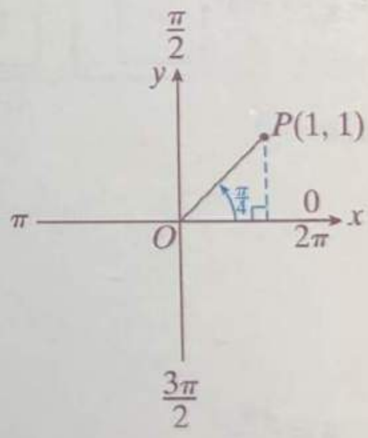


$\sin\theta = \frac{1}{2}$

$\cos\theta = \frac{\sqrt{3}}{2}$

$\tan\theta = \frac{\sqrt{3}}{3}$

(2) $\theta = \frac{\pi}{4}$

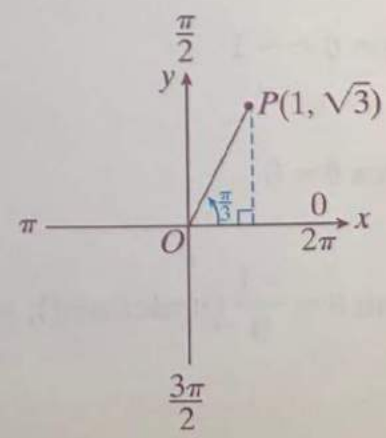


$\sin\theta = \frac{\sqrt{2}}{2}$

$\cos\theta = \frac{\sqrt{2}}{2}$

$\tan\theta = 1$

(3) $\theta = \frac{\pi}{3}$



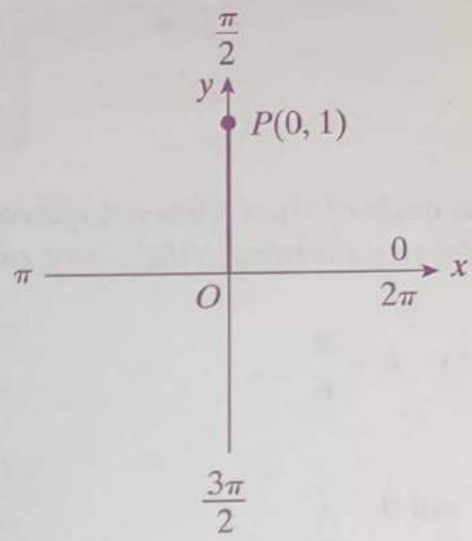
$\sin\theta = \frac{\sqrt{3}}{2}$

$\cos\theta = \frac{1}{2}$

$\tan\theta = \sqrt{3}$

M 46 b

(4) $\theta = \frac{\pi}{2}$

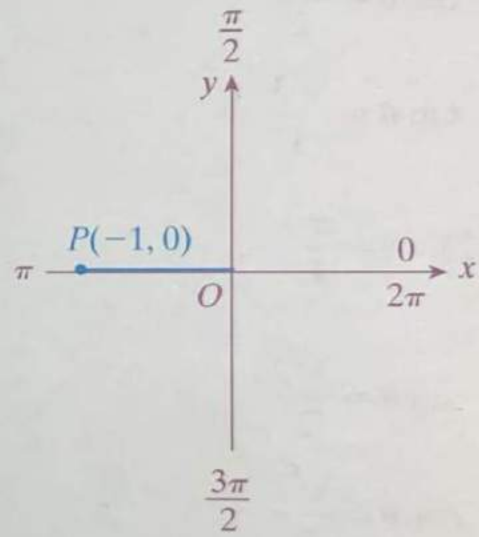


$\sin\theta = 1$

$\cos\theta = 0$

$\tan\theta = \frac{1}{0}$ (undefined); ∞

(5) $\theta = \pi$

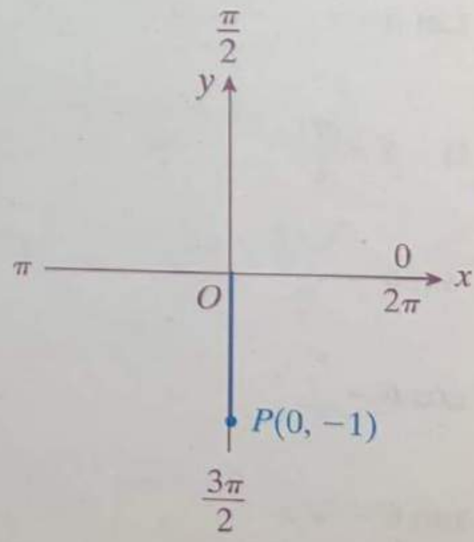


$\sin\theta = 0$

$\cos\theta = -1$

$\tan\theta = 0$

(6) $\theta = \frac{3\pi}{2}$



$\sin\theta = -1$

$\cos\theta = 0$

$\tan\theta = \frac{-1}{0}$ (undefined); ∞

Trigonometric Functions 5

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	1	2~3	4~5	6~

1. Fill in the blanks to complete the table below.

Trigonometric Functions of Special Angles

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
	$0°, 0$	$30°, \frac{\pi}{6}$	$45°, \boxed{\frac{\pi}{4}}$	$60°, \boxed{\frac{\pi}{3}}$	$90°, \boxed{\frac{\pi}{2}}$	$180°, \boxed{\pi}$	$270°, \boxed{\frac{3\pi}{2}}$	$360°, \boxed{2\pi}$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\boxed{\frac{\sqrt{3}}{2}}$	$\boxed{1}$	$\boxed{0}$	$\boxed{-1}$	$\boxed{0}$
cos	1	$\boxed{\frac{\sqrt{3}}{2}}$	$\boxed{\frac{\sqrt{2}}{2}}$	$\boxed{\frac{1}{2}}$	$\boxed{0}$	-1	$\boxed{0}$	$\boxed{1}$
tan	$\boxed{0}$	$\boxed{\frac{\sqrt{3}}{3}}$	$\boxed{1}$	$\boxed{\sqrt{3}}$	∞	$\boxed{0}$	$\boxed{\infty}$	$\boxed{0}$

2. Evaluate the following expressions.

(1) $\sin\frac{\pi}{3} + \cos\frac{\pi}{4} = \frac{\sqrt{3}+\sqrt{2}}{2}$

(2) $\cos\pi \tan\frac{\pi}{3} = -1 \cdot \sqrt{3} = -\sqrt{3}$

(3) $$\frac{\cos\frac{\pi}{6}\sin\frac{\pi}{4}}{\cos^2\frac{\pi}{3}\tan\frac{\pi}{4}} = \frac{\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}}{\left(\frac{1}{2}\right)^2\cdot 1} = \frac{\frac{\sqrt{6}}{4}}{\frac{1}{4}} = \sqrt{6}$$

M 47 b

(4) $$\frac{\cos\frac{\pi}{4}+\cos\frac{\pi}{3}}{\sin\frac{\pi}{6}\tan\frac{\pi}{4}}=\frac{\frac{\sqrt{2}}{2}+\frac{1}{2}}{\frac{1}{2}\cdot 1}=\frac{\frac{\sqrt{2}+1}{2}}{\frac{1}{2}}=\sqrt{2}+1$$

(5) $$\tan\frac{\pi}{6}\tan\frac{\pi}{3}-\cos 2\pi\cos\pi+2\sin\frac{\pi}{3}$$

$$=\frac{\sqrt{3}}{3}\cdot\sqrt{3}-1\cdot(-1)+2\cdot\frac{\sqrt{3}}{2}=2+\sqrt{3}$$

(6) $$\frac{\sin^2\frac{\pi}{4}\tan^2\frac{\pi}{3}}{1+\sin\frac{\pi}{6}\sin^2\frac{\pi}{3}}=\frac{\left(\frac{\sqrt{2}}{2}\right)^2(\sqrt{3})^2}{1+\frac{1}{2}\cdot\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{\frac{1}{2}\cdot 3}{1+\frac{1}{2}\cdot\frac{3}{4}}=\frac{\frac{3}{2}}{\frac{11}{8}}=\frac{12}{11}$$

(7) $$\frac{\sin\frac{3\pi}{2}}{\tan\frac{\pi}{3}}+\frac{\tan\frac{\pi}{6}}{\sin\frac{\pi}{2}}=\frac{-1}{\sqrt{3}}+\frac{\frac{\sqrt{3}}{3}}{1}=-\frac{\sqrt{3}}{3}+\frac{\sqrt{3}}{3}=0$$

(8) $$\frac{\cos^2\frac{\pi}{4}-\cos^2\frac{\pi}{3}}{\cos^2 2\pi-\cos^2\frac{\pi}{6}}=\frac{\left(\frac{\sqrt{2}}{2}\right)^2-\left(\frac{1}{2}\right)^2}{1^2-\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{\frac{1}{2}-\frac{1}{4}}{1-\frac{3}{4}}=\frac{\frac{1}{4}}{\frac{1}{4}}=1$$

(9) $$\sin\frac{3\pi}{2}\tan\pi+\cos\pi\sin\frac{\pi}{2}+\tan^2\frac{\pi}{3}\tan\frac{\pi}{4}$$

$$=(-1)(0)+(-1)(1)+\left(\sqrt{3}\right)^2(1)=0-1+3=2$$

(10) $$\frac{\tan\frac{\pi}{6}\cos^2\frac{\pi}{6}}{\tan\frac{\pi}{3}\sin^2\frac{\pi}{6}}=\frac{\frac{\sqrt{3}}{3}\cdot\left(\frac{\sqrt{3}}{2}\right)^2}{\sqrt{3}\cdot\left(\frac{1}{2}\right)^2}=\frac{\frac{\sqrt{3}}{3}\cdot\frac{3}{4}}{\sqrt{3}\cdot\frac{1}{4}}=1$$

Trigonometric Functions 5

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

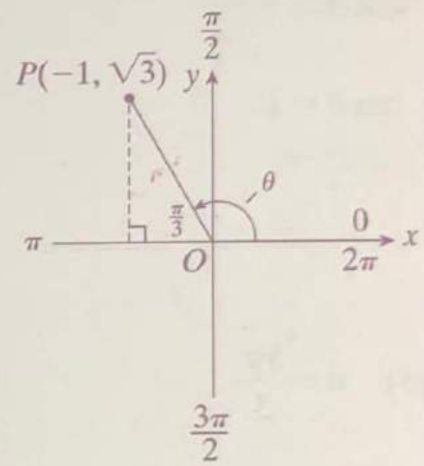
In each of the following exercises, draw the diagram, and use it to evaluate the trigonometric functions of the given angle.

(1) $\theta = \frac{2\pi}{3}$

$\sin\theta = \frac{\sqrt{3}}{2}$

$\cos\theta = -\frac{1}{2}$

$\tan\theta = -\sqrt{3}$

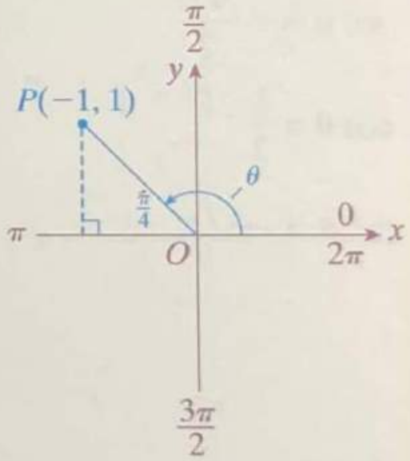


(2) $\theta = \frac{3\pi}{4}$

$\sin\theta = \frac{\sqrt{2}}{2}$

$\cos\theta = -\frac{\sqrt{2}}{2}$

$\tan\theta = -1$

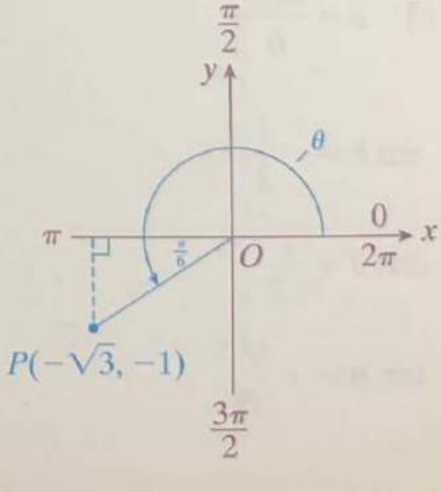


(3) $\theta = \frac{7\pi}{6}$

$\sin\theta = -\frac{1}{2}$

$\cos\theta = -\frac{\sqrt{3}}{2}$

$\tan\theta = \frac{\sqrt{3}}{3}$

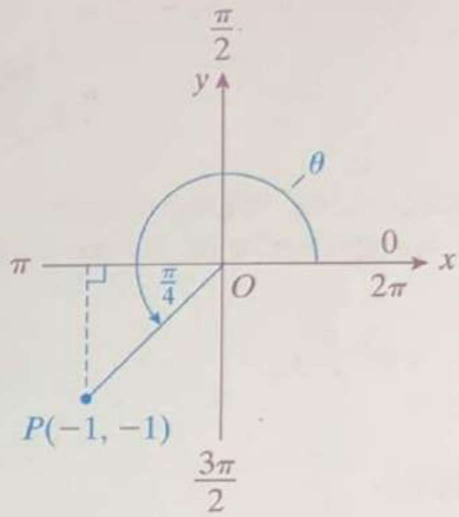


(4) $\theta = \frac{5\pi}{4}$

$\sin\theta = -\frac{\sqrt{2}}{2}$

$\cos\theta = -\frac{\sqrt{2}}{2}$

$\tan\theta = 1$

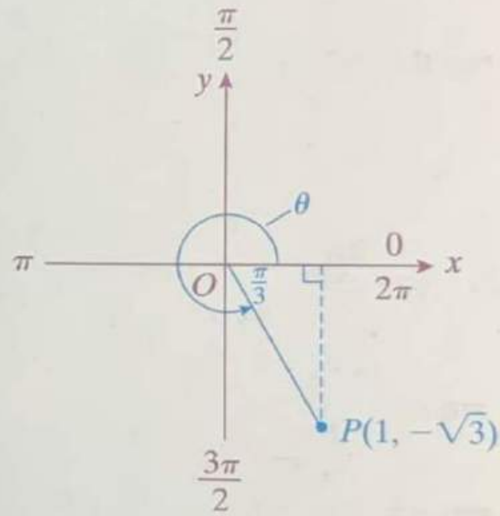


(5) $\theta = \frac{5\pi}{3}$

$\sin\theta = -\frac{\sqrt{3}}{2}$

$\cos\theta = \frac{1}{2}$

$\tan\theta = -\sqrt{3}$

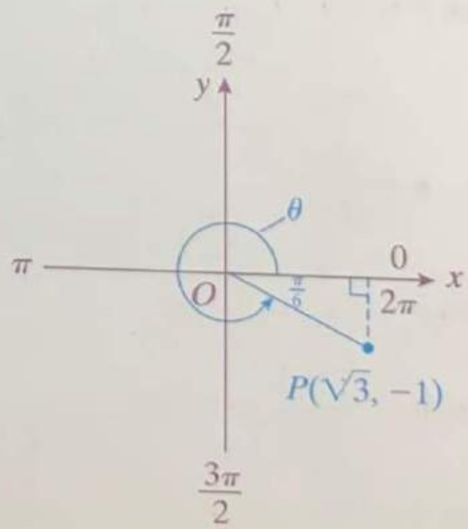


(6) $\theta = \frac{11\pi}{6}$

$\sin\theta = -\frac{1}{2}$

$\cos\theta = \frac{\sqrt{3}}{2}$

$\tan\theta = -\frac{\sqrt{3}}{3}$



KUMON

M 49 a Trigonometric Functions 5

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

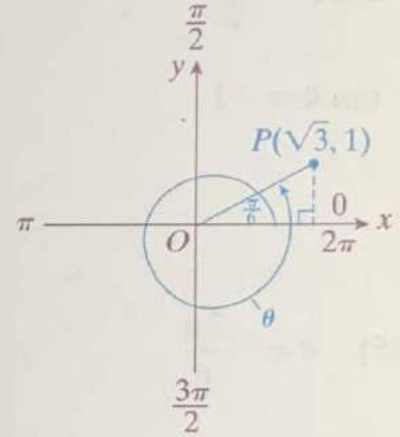
In each of the following exercises, draw the diagram, and use it to evaluate the trigonometric functions of the given angle.

(1) $\theta = \dfrac{13\pi}{6}$

$\sin\theta = \dfrac{1}{2}$

$\cos\theta = \dfrac{\sqrt{3}}{2}$

$\tan\theta = \dfrac{\sqrt{3}}{3}$

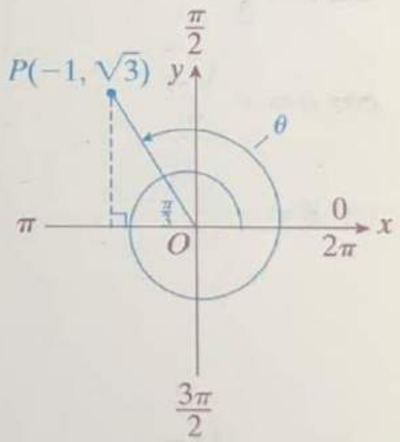


(2) $\theta = \dfrac{8\pi}{3}$

$\sin\theta = \dfrac{\sqrt{3}}{2}$

$\cos\theta = -\dfrac{1}{2}$

$\tan\theta = -\sqrt{3}$

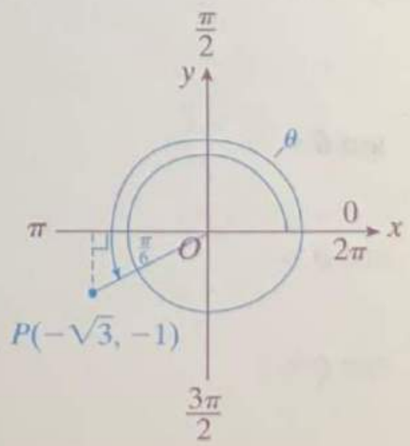


(3) $\theta = \dfrac{19\pi}{6}$

$\sin\theta = -\dfrac{1}{2}$

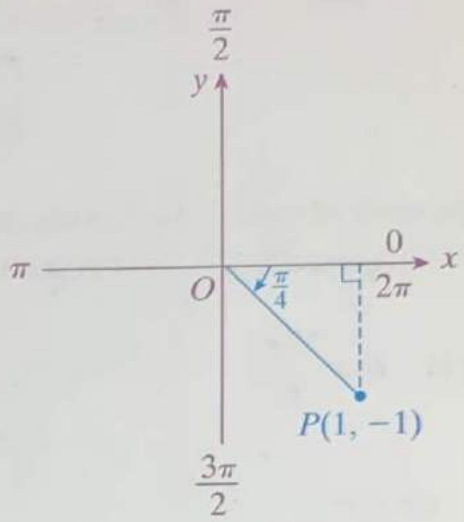
$\cos\theta = -\dfrac{\sqrt{3}}{2}$

$\tan\theta = \dfrac{\sqrt{3}}{3}$



M 49 b

(4) $\theta = -\dfrac{\pi}{4}$

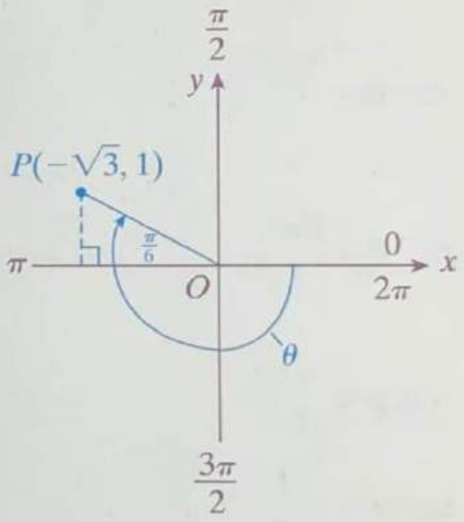


$\sin\theta = -\dfrac{\sqrt{2}}{2}$

$\cos\theta = \dfrac{\sqrt{2}}{2}$

$\tan\theta = -1$

(5) $\theta = -\dfrac{7\pi}{6}$

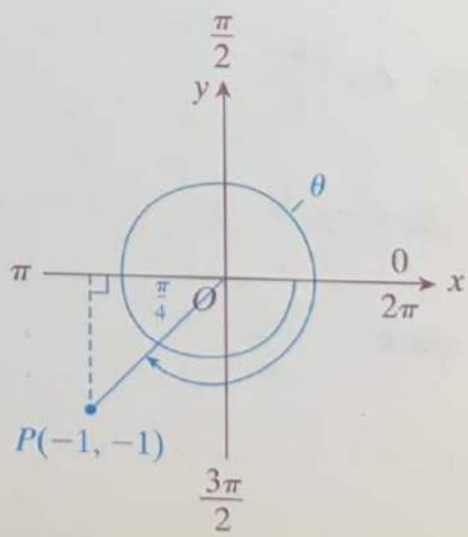


$\sin\theta = \dfrac{1}{2}$

$\cos\theta = -\dfrac{\sqrt{3}}{2}$

$\tan\theta = -\dfrac{\sqrt{3}}{3}$

(6) $\theta = -\dfrac{11\pi}{4}$



$\sin\theta = -\dfrac{\sqrt{2}}{2}$

$\cos\theta = -\dfrac{\sqrt{2}}{2}$

$\tan\theta = 1$

KUMON

M 50 a

Trigonometric Functions 5

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	1~2	3~4	5~6	7~

Evaluate each of the following expressions.

(1) $\tan\frac{3}{4}\pi = -1$

(2) $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

(3) $\cos 2\pi = 1$

(4) $\sin(-\pi) = 0$

(5) $\cos\frac{4}{3}\pi = -\frac{1}{2}$

(6) $\tan\left(-\frac{2}{3}\pi\right) = \sqrt{3}$

(7) $\sin\frac{7}{4}\pi = -\frac{\sqrt{2}}{2}$

(8) $\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

(9) $\sin\frac{15}{2}\pi = -1$

M 50 b

(10) $\cos\left(-\frac{\pi}{2}\right)=0$

(11) $\cos\frac{3}{4}\pi=-\frac{\sqrt{2}}{2}$

(12) $\tan\frac{7}{6}\pi=\frac{\sqrt{3}}{3}$

(13) $\sin\frac{7}{2}\pi=-1$

(14) $\cos\left(-\frac{7}{4}\pi\right)=\frac{\sqrt{2}}{2}$

(15) $\tan\left(-\frac{4}{3}\pi\right)=-\sqrt{3}$

(16) $\sin\frac{5}{3}\pi=-\frac{\sqrt{3}}{2}$

(17) $\cos\frac{10}{3}\pi=-\frac{1}{2}$

(18) $\sin\left(-\frac{3}{2}\pi\right)=1$

(19) $\cos\frac{11}{6}\pi=\frac{\sqrt{3}}{2}$

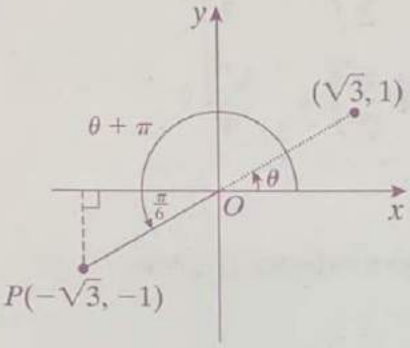
(20) $\sin\left(-\frac{13}{3}\pi\right)=-\frac{\sqrt{3}}{2}$

Trigonometric Functions 6

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. Given $\theta = \dfrac{\pi}{6}$, draw the diagram, and use it to evaluate the sin, cos and tan of $(\theta + \pi)$.

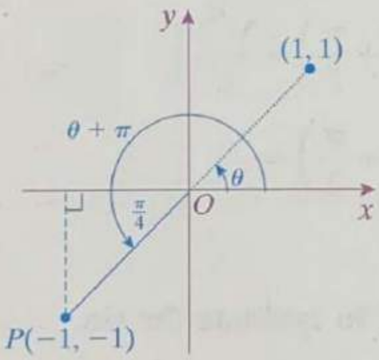


$\sin(\theta + \pi) = -\dfrac{1}{2}$

$\cos(\theta + \pi) = -\dfrac{\sqrt{3}}{2}$

$\tan(\theta + \pi) = \dfrac{\sqrt{3}}{3}$

2. Given $\theta = \dfrac{\pi}{4}$, draw the diagram, and use it to evaluate the sin, cos and tan of $(\theta + \pi)$.

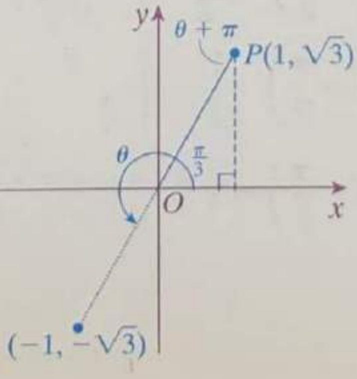


$\sin(\theta + \pi) = -\dfrac{\sqrt{2}}{2}$

$\cos(\theta + \pi) = -\dfrac{\sqrt{2}}{2}$

$\tan(\theta + \pi) = 1$

3. Given $\theta = \dfrac{4\pi}{3}$, draw the diagram, and use it to evaluate the sin, cos and tan of $(\theta + \pi)$.



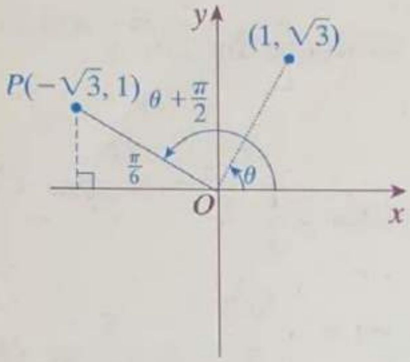
$\sin(\theta + \pi) = \dfrac{\sqrt{3}}{2}$

$\cos(\theta + \pi) = \dfrac{1}{2}$

$\tan(\theta + \pi) = \sqrt{3}$

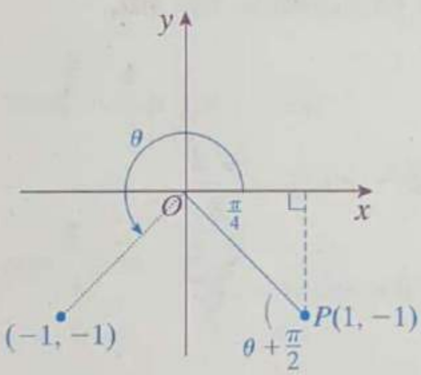
M 51 b

4. Given $\theta = \frac{\pi}{3}$, draw the diagram, and use it to evaluate the sin, cos and tan of $\left(\theta + \frac{\pi}{2}\right)$.



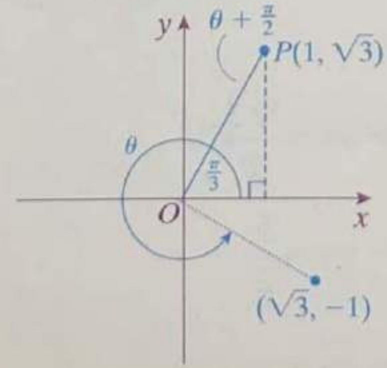
$$\sin\left(\theta + \frac{\pi}{2}\right) = \frac{1}{2}$$
$$\cos\left(\theta + \frac{\pi}{2}\right) = -\frac{\sqrt{3}}{2}$$
$$\tan\left(\theta + \frac{\pi}{2}\right) = -\frac{\sqrt{3}}{3}$$

5. Given $\theta = \frac{5\pi}{4}$, draw the diagram, and use it to evaluate the sin, cos and tan of $\left(\theta + \frac{\pi}{2}\right)$.



$$\sin\left(\theta + \frac{\pi}{2}\right) = -\frac{\sqrt{2}}{2}$$
$$\cos\left(\theta + \frac{\pi}{2}\right) = \frac{\sqrt{2}}{2}$$
$$\tan\left(\theta + \frac{\pi}{2}\right) = -1$$

6. Given $\theta = \frac{11\pi}{6}$, draw the diagram, and use it to evaluate the sin, cos and tan of $\left(\theta + \frac{\pi}{2}\right)$.



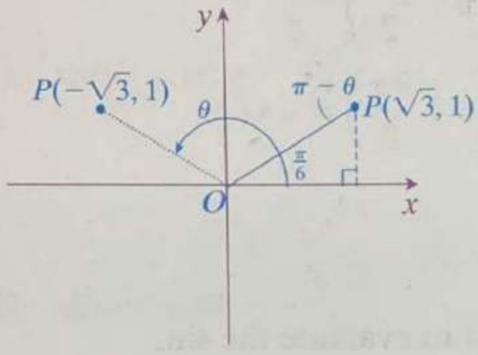
$$\sin\left(\theta + \frac{\pi}{2}\right) = \frac{\sqrt{3}}{2}$$
$$\cos\left(\theta + \frac{\pi}{2}\right) = \frac{1}{2}$$
$$\tan\left(\theta + \frac{\pi}{2}\right) = \sqrt{3}$$

M 52 a   Trigonometric Functions 6

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. Given $\theta = \dfrac{5\pi}{6}$, draw the diagram, and use it to evaluate the sin, cos and tan of $(\pi - \theta)$.

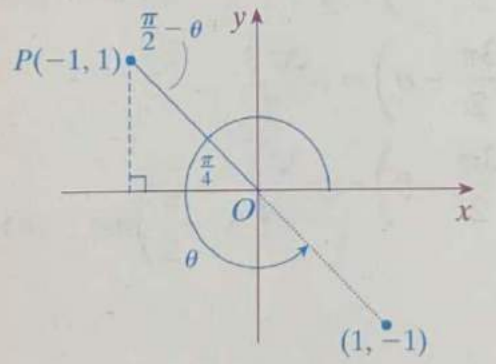


$\sin(\pi - \theta) = \dfrac{1}{2}$

$\cos(\pi - \theta) = \dfrac{\sqrt{3}}{2}$

$\tan(\pi - \theta) = \dfrac{\sqrt{3}}{3}$

2. Given $\theta = \dfrac{7\pi}{4}$, draw the diagram, and use it to evaluate the sin, cos and tan of $\left(\dfrac{\pi}{2} - \theta\right)$.

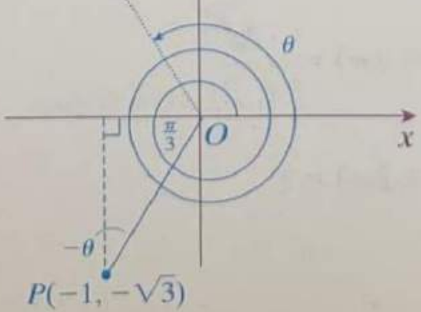


$\sin\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{\sqrt{2}}{2}$

$\cos\left(\dfrac{\pi}{2} - \theta\right) = -\dfrac{\sqrt{2}}{2}$

$\tan\left(\dfrac{\pi}{2} - \theta\right) = -1$

3. Given $\theta = \dfrac{14\pi}{3}$, draw the diagram, and use it to evaluate the sin, cos and tan of $(-\theta)$.



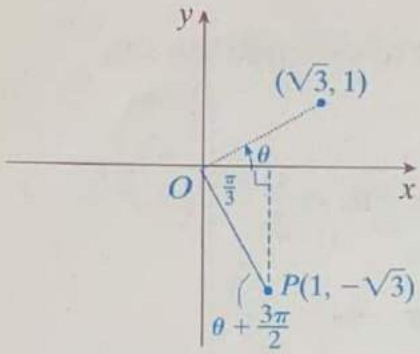
$\sin(-\theta) = -\dfrac{\sqrt{3}}{2}$

$\cos(-\theta) = -\dfrac{1}{2}$

$\tan(-\theta) = \sqrt{3}$

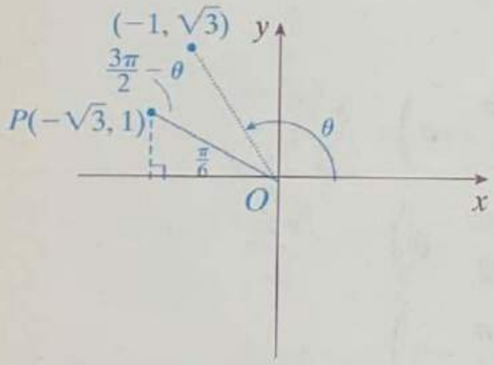
M 52 b

4. Given $\theta = \frac{\pi}{6}$, draw the diagram, and use it to evaluate the sin, cos and tan of $\left(\theta + \frac{3\pi}{2}\right)$.



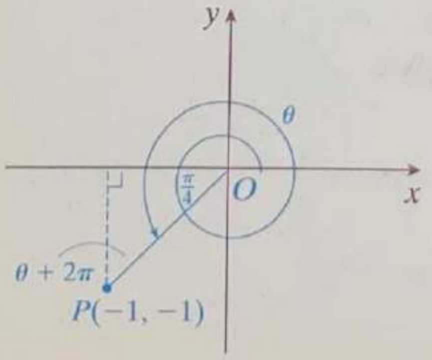
$$\sin\left(\theta + \frac{3\pi}{2}\right) = -\frac{\sqrt{3}}{2}$$
$$\cos\left(\theta + \frac{3\pi}{2}\right) = \frac{1}{2}$$
$$\tan\left(\theta + \frac{3\pi}{2}\right) = -\sqrt{3}$$

5. Given $\theta = \frac{2\pi}{3}$, draw the diagram, and use it to evaluate the sin, cos and tan of $\left(\frac{3\pi}{2} - \theta\right)$.



$$\sin\left(\frac{3\pi}{2} - \theta\right) = \frac{1}{2}$$
$$\cos\left(\frac{3\pi}{2} - \theta\right) = -\frac{\sqrt{3}}{2}$$
$$\tan\left(\frac{3\pi}{2} - \theta\right) = -\frac{\sqrt{3}}{3}$$

6. Given $\theta = \frac{13\pi}{4}$, draw the diagram, and use it to evaluate the sin, cos and tan of $(\theta + 2\pi)$.



$$\sin(\theta + 2\pi) = -\frac{\sqrt{2}}{2}$$
$$\cos(\theta + 2\pi) = -\frac{\sqrt{2}}{2}$$
$$\tan(\theta + 2\pi) = 1$$

KUMON

M 53 a

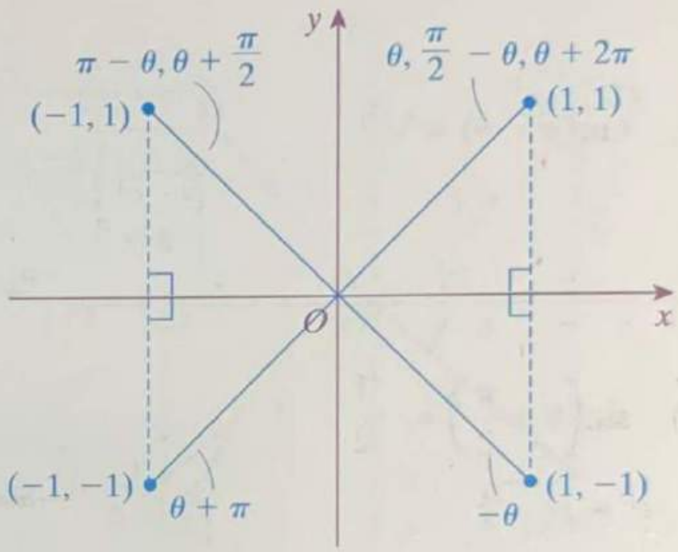
Trigonometric Functions 6

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	1	2	3	4~

1. Given that $\theta = \frac{\pi}{4}$, draw the diagram, and use it to evaluate each of the following given trigonometric expressions.

(1) $\sin(\theta + \pi) = -\frac{\sqrt{2}}{2}$



Sketch all the diagrams on this coordinate grid.

(2) $\cos(\pi - \theta) = -\frac{\sqrt{2}}{2}$

(3) $\tan\left(\theta + \frac{\pi}{2}\right) = -1$

(4) $\sin\left(\frac{\pi}{2} - \theta\right) = \frac{\sqrt{2}}{2}$

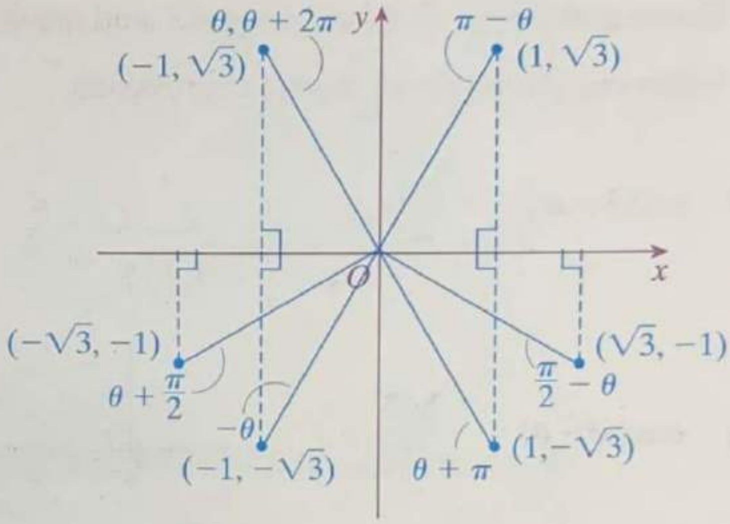
(5) $\cos(-\theta) = \frac{\sqrt{2}}{2}$

(6) $\tan(\theta + 2\pi) = 1$

M 53 b

2. Given that $\theta = \dfrac{8\pi}{3}$, draw the diagram, and use it to evaluate each of the following given trigonometric expressions.

(1) $\cos(\theta + \pi) = \dfrac{1}{2}$



Sketch all the diagrams on this coordinate grid.

(2) $\tan(\pi - \theta) = \sqrt{3}$

(3) $\sin\left(\theta + \dfrac{\pi}{2}\right) = -\dfrac{1}{2}$

(4) $\cos\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{\sqrt{3}}{2}$

(5) $\tan(-\theta) = \sqrt{3}$

(6) $\sin(\theta + 2\pi) = \dfrac{\sqrt{3}}{2}$

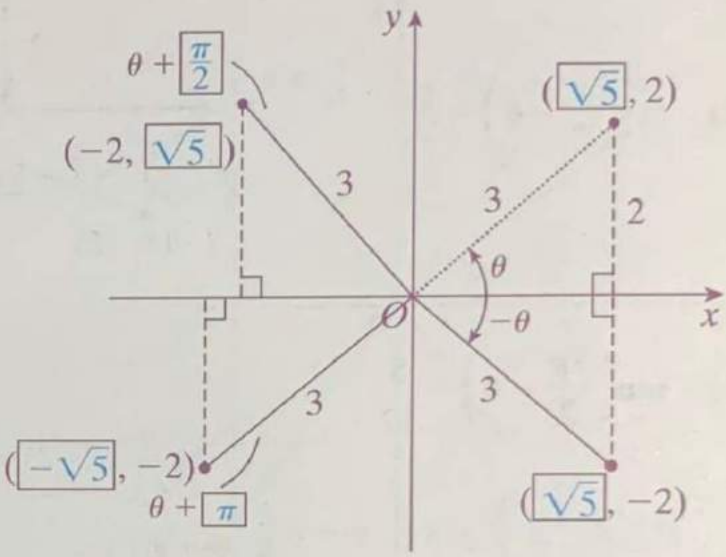
KUMON

M 54 a Trigonometric Functions 6

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	1	2	3	4~

1. Given that $0<\theta<\frac{\pi}{2}$, and $\sin\theta=\frac{2}{3}$, draw the diagram, and use it to evaluate each of the following trigonometric expressions.



(1) $\sin(-\theta)=-\frac{2}{3}$

(2) $\cos(\theta+\pi)=-\frac{\sqrt{5}}{3}$

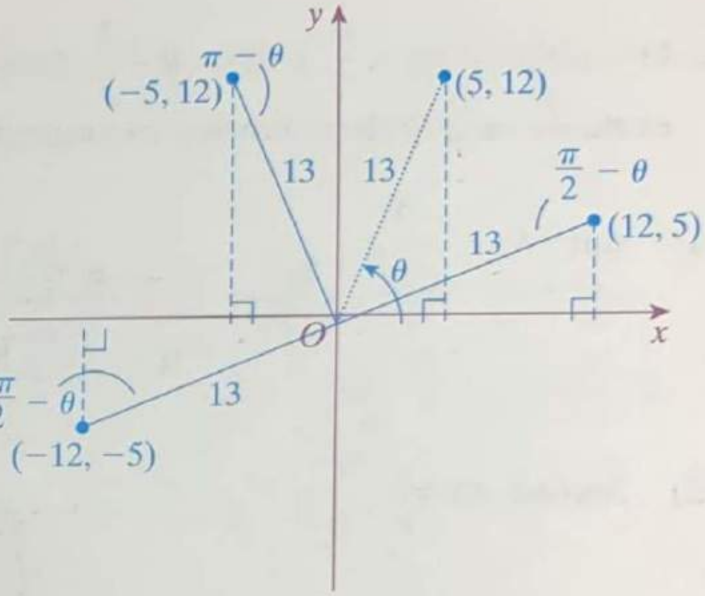
(3) $\tan\left(\theta+\frac{\pi}{2}\right)=-\frac{\sqrt{5}}{2}$

(4) $\cos(-\theta)=\frac{\sqrt{5}}{3}$

(5) $\tan(\theta+\pi)=\frac{2\sqrt{5}}{5}$

(6) $\sin\left(\theta+\frac{\pi}{2}\right)=\frac{\sqrt{5}}{3}$

2. Given that $0 < \theta < \frac{\pi}{2}$, and $\cos\theta = \frac{5}{13}$, draw the diagram, and use it to evaluate each of the following trigonometric expressions.



(1) $\sin(\pi - \theta) = \frac{12}{13}$

(2) $\cos\left(\frac{\pi}{2} - \theta\right) = \frac{12}{13}$

(3) $\tan\left(\frac{3\pi}{2} - \theta\right) = \frac{5}{12}$

(4) $\cos(\pi - \theta) = -\frac{5}{13}$

(5) $\tan\left(\frac{\pi}{2} - \theta\right) = \frac{5}{12}$

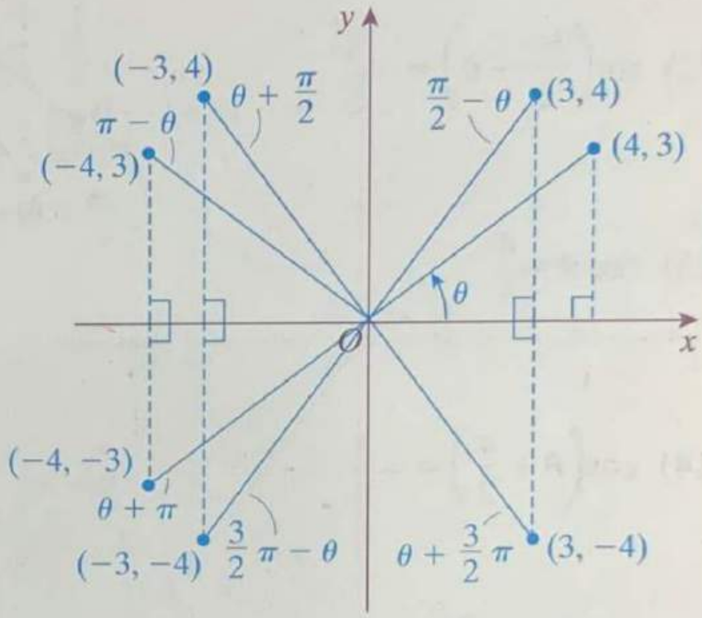
(6) $\sin\left(\frac{3\pi}{2} - \theta\right) = -\frac{5}{13}$

M 55 a Trigonometric Functions 6

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	1	2~3	4~5	6~

Given that $0<\theta<\frac{\pi}{2}$, and $\sin\theta=\frac{3}{5}$, draw the diagram, and use it to evaluate the following trigonometric expressions.



(1) $\sin\left(\theta+\frac{\pi}{2}\right)=\frac{4}{5}$

(2) $\sin(\theta+\pi)=-\frac{3}{5}$

(3) $\sin\left(\theta+\frac{3}{2}\pi\right)=-\frac{4}{5}$

(4) $\sin\left(\frac{\pi}{2}-\theta\right)=\frac{4}{5}$

(5) $\cos(\theta+\pi)=-\frac{4}{5}$

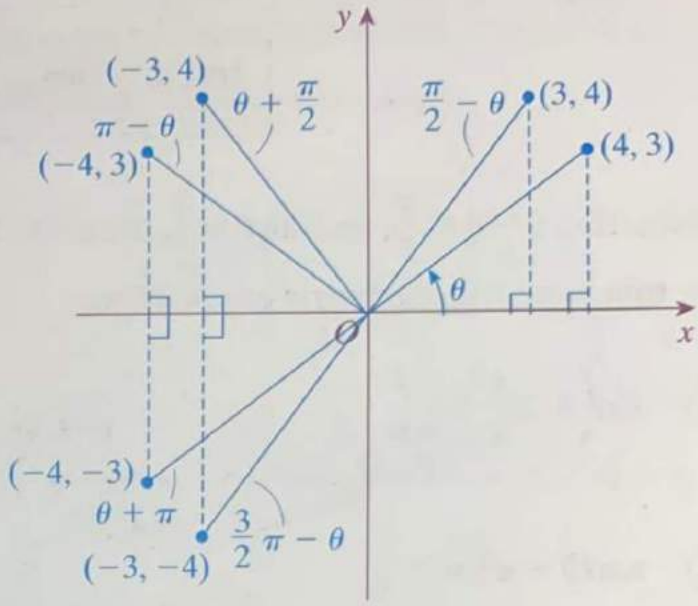
(6) $\cos\left(\theta+\frac{3}{2}\pi\right)=\frac{3}{5}$

(7) $\cos\left(\frac{\pi}{2}-\theta\right)=\frac{3}{5}$

(8) $\cos(\pi-\theta)=-\frac{4}{5}$

(9) $\cos\left(\frac{3}{2}\pi-\theta\right)=-\frac{3}{5}$

M 55 b



(10) $\tan\theta = \frac{3}{4}$

(11) $\sin(\pi - \theta) = \frac{3}{5}$

(12) $\sin\left(\frac{3\pi}{2} - \theta\right) = -\frac{4}{5}$

(13) $\cos\theta = \frac{4}{5}$

(14) $\cos\left(\theta + \frac{\pi}{2}\right) = -\frac{3}{5}$

(15) $\tan\left(\theta + \frac{\pi}{2}\right) = -\frac{4}{3}$

(16) $\tan(\theta + \pi) = \frac{3}{4}$

(17) $\tan\left(\frac{\pi}{2} - \theta\right) = \frac{4}{3}$

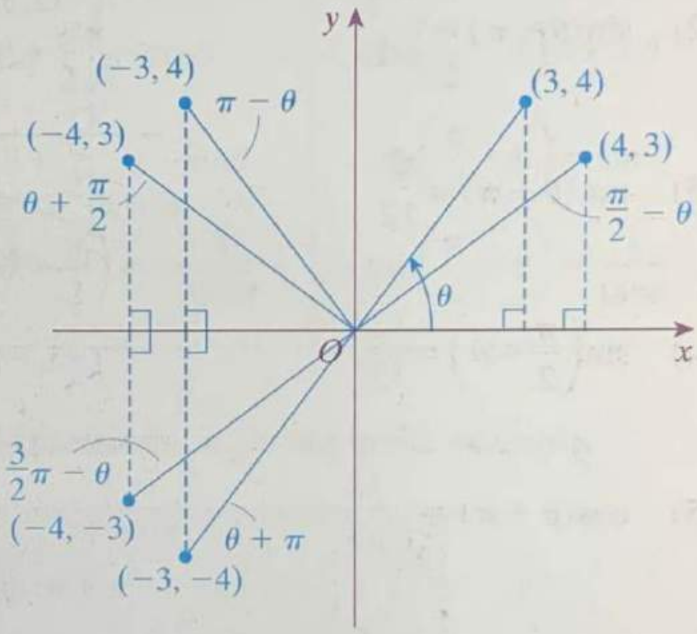
(18) $\tan(\pi - \theta) = -\frac{3}{4}$

M 56 a Trigonometric Functions 6

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	1	2~3	4~5	6~

1. Given that $0<\theta<\frac{\pi}{2}$, and $\cos\theta=\frac{3}{5}$, draw the diagram, and use it to evaluate the following trigonometric expressions.



(1) $\cos\left(\theta+\frac{\pi}{2}\right)=-\frac{4}{5}$

(2) $\cos\left(\frac{\pi}{2}-\theta\right)=\frac{4}{5}$

(3) $\cos(\pi-\theta)=-\frac{3}{5}$

(4) $\sin\left(\theta+\frac{\pi}{2}\right)=\frac{3}{5}$

(5) $\sin(\theta+\pi)=-\frac{4}{5}$

(6) $\sin\left(\frac{3}{2}\pi-\theta\right)=-\frac{3}{5}$

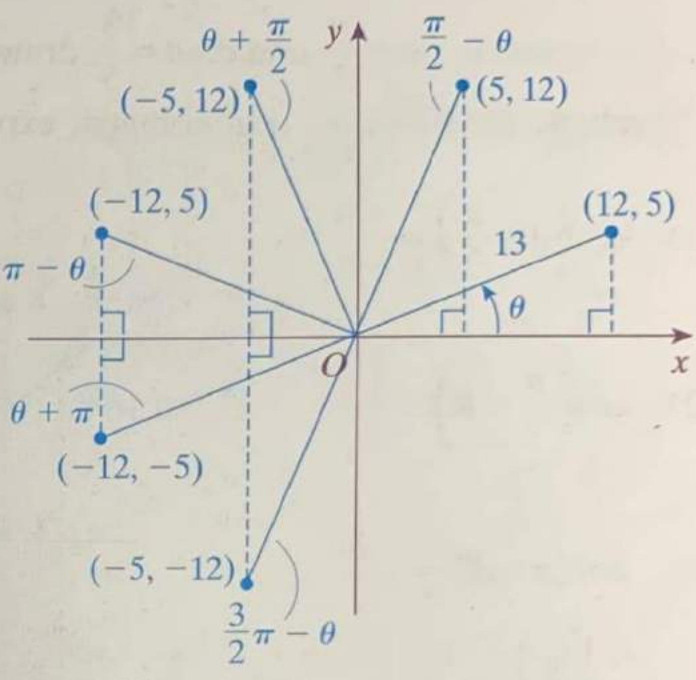
(7) $\tan\left(\theta+\frac{\pi}{2}\right)=-\frac{3}{4}$

(8) $\tan(\theta+\pi)=\frac{4}{3}$

(9) $\tan(\pi-\theta)=-\frac{4}{3}$

M 56 b

2. Given that $0<\theta<\frac{\pi}{2}$, and $\sin\theta=\frac{5}{13}$, draw the diagram, and use it to evaluate the following trigonometric expressions.



(1) $\sin\left(\theta+\frac{\pi}{2}\right)=\frac{12}{13}$

(2) $\sin(\theta+\pi)=-\frac{5}{13}$

(3) $\tan(\theta+\pi)=\frac{5}{12}$

(4) $\sin\left(\frac{\pi}{2}-\theta\right)=\frac{12}{13}$

(5) $\cos(\theta+\pi)=-\frac{12}{13}$

(6) $\tan(\pi-\theta)=-\frac{5}{12}$

(7) $\cos\left(\frac{\pi}{2}-\theta\right)=\frac{5}{13}$

(8) $\cos(\pi-\theta)=-\frac{12}{13}$

(9) $\cos\left(\frac{3}{2}\pi-\theta\right)=-\frac{5}{13}$

Trigonometric Functions 6

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Review Formulas	$\sin(\theta+\pi)=-\sin\theta$ $\cos(\theta+\pi)=-\cos\theta$ $\tan(\theta+\pi)=\tan\theta$	$\sin(\pi-\theta)=\sin\theta$ $\cos(\pi-\theta)=-\cos\theta$ $\tan(\pi-\theta)=-\tan\theta$
$\sin(-\theta)=-\sin\theta$ $\cos(-\theta)=\cos\theta$ $\tan(-\theta)=-\tan\theta$	$\sin\left(\theta+\frac{\pi}{2}\right)=\cos\theta$ $\cos\left(\theta+\frac{\pi}{2}\right)=-\sin\theta$ $\tan\left(\theta+\frac{\pi}{2}\right)=-\frac{1}{\tan\theta}$	$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$ $\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$ $\tan\left(\frac{\pi}{2}-\theta\right)=\frac{1}{\tan\theta}$

Simplify each of the following expressions, as shown in the example.

Ex.
$$\sin\theta+\sin\left(\theta+\frac{\pi}{2}\right)+\sin(\theta+\pi)+\sin\left(\theta+\frac{3}{2}\pi\right)$$
$$=\sin\theta+\cos\theta-\sin\theta-\cos\theta$$
$$=0$$

(1) $\cos\theta+\cos\left(\theta+\frac{\pi}{2}\right)+\cos(\theta+\pi)+\cos\left(\theta+\frac{3}{2}\pi\right)$
$$=\cos\theta-\sin\theta-\cos\theta+\sin\theta$$
$$=0$$

(2) $\cos\left(\frac{\pi}{2}-\theta\right)\sin(\pi-\theta)-\sin\left(\frac{\pi}{2}-\theta\right)\cos(\pi-\theta)$
$$=\sin\theta\sin\theta-\cos\theta(-\cos\theta)$$
$$=\sin^2\theta+\cos^2\theta$$
$$=1$$

M 57 b

(3) $\sin^2(\theta+\pi)+\sin^2\left(\theta-\frac{\pi}{2}\right)$

$=(-\sin\theta)^2+(\boxed{-\cos\theta})^2$

$=\sin^2\theta+\cos^2\theta$

$=1$

(4) $\cos^2(\theta+\pi)+\cos^2\left(\theta-\frac{\pi}{2}\right)$

$=(-\cos\theta)^2+\sin^2\theta$

$=\cos^2\theta+\sin^2\theta$

$=1$

(5) $\dfrac{\cos(\pi+\theta)}{1+\cos\left(\frac{\pi}{2}+\theta\right)}\times\dfrac{\cos(\pi-\theta)}{1+\cos\left(\frac{\pi}{2}-\theta\right)}$

$=\dfrac{-\cos\theta}{1-\sin\theta}\cdot\dfrac{-\cos\theta}{1+\sin\theta}$

$=\dfrac{\cos^2\theta}{1-\sin^2\theta}$

$=\dfrac{\cos^2\theta}{\cos^2\theta}$

$=1$

M 58 a

Trigonometric Functions 6

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

Simplify each of the following expressions.

(1) $\dfrac{\sin(-\theta)}{\sin(\theta+\pi)} - \tan\left(\theta+\dfrac{\pi}{2}\right)\tan\theta + \dfrac{\cos(2\pi-\theta)}{\sin\left(\theta+\dfrac{\pi}{2}\right)}$

$= \dfrac{-\sin\theta}{-\sin\theta} - \left(-\dfrac{1}{\tan\theta}\right)\tan\theta + \dfrac{\cos\theta}{\cos\theta}$

$= 1+1+1$

$= 3$

(2) $\dfrac{\sin\left(\theta+\dfrac{\pi}{2}\right)}{\cos\left(\theta-\dfrac{\pi}{2}\right)} \times \tan(\theta+\pi)\tan(\theta-\pi) \times \dfrac{\sin\left(\theta-\dfrac{3}{2}\pi\right)}{\cos\left(\theta+\dfrac{3}{2}\pi\right)}$

$= \dfrac{\cos\theta}{\sin\theta} \cdot \tan^2\theta \cdot \dfrac{\cos\theta}{\sin\theta}$

$= \dfrac{\cos\theta}{\sin\theta} \cdot \dfrac{\sin^2\theta}{\cos^2\theta} \cdot \dfrac{\cos\theta}{\sin\theta}$

$= 1$

M 58 b

(3) $\dfrac{(a^2-b^2)\tan(\pi+\theta)}{\tan(\pi-\theta)}-(a^2+b^2)\tan\left(\dfrac{\pi}{2}-\theta\right)\tan(\pi-\theta)$

$= -\dfrac{(a^2-b^2)\tan\theta}{\tan\theta}+\dfrac{(a^2+b^2)\tan\theta}{\tan\theta}$

$= -(a^2-b^2)+(a^2+b^2)$

$= 2b^2$

(4) $\dfrac{\sin(\pi+\theta)\tan^2(\pi-\theta)}{\cos\left(\dfrac{3}{2}\pi+\theta\right)}-\dfrac{\sin\left(\dfrac{3}{2}\pi-\theta\right)}{\sin\left(\dfrac{\pi}{2}+\theta\right)}+\tan^2(\pi+\theta)$

$= \dfrac{(-\sin\theta)(-\tan\theta)^2}{\sin\theta}-\dfrac{(-\cos\theta)}{\cos\theta}+\tan^2\theta$

$= -\tan^2\theta+1+\tan^2\theta$

$= 1$

M 59 a

Trigonometric Functions 6

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

Simplify each of the following expressions.

(1) $\sin^2\left(A+\frac{\pi}{4}\right)+\sin^2\left(A-\frac{\pi}{4}\right)$ $\left(\text{Let } A+\frac{\pi}{4}=\theta\right)$

$=\sin^2\theta+\sin^2\left(\theta-\frac{\pi}{2}\right)$

$=\sin^2\theta+(-\cos\theta)^2$

$=1$

$\left(\begin{aligned}\text{Since } A+\frac{\pi}{4}&=\theta,\\ A-\frac{\pi}{4}&=\theta-\frac{\pi}{2}\end{aligned}\right)$

(2) $\tan x-\tan\left(x-\frac{\pi}{2}\right)-\dfrac{1}{\cos x\cos\left(x-\frac{\pi}{2}\right)}$

$=\tan x+\dfrac{1}{\tan x}-\dfrac{1}{\cos x\sin x}$

$=\dfrac{\sin x}{\cos x}+\dfrac{\cos x}{\sin x}-\dfrac{1}{\cos x\sin x}$

$=\dfrac{\sin^2 x+\cos^2 x-1}{\sin x\cos x}$

$=0$

(3) $$\frac{\sin(\pi+\theta)\tan^2(\pi-\theta)}{\cos\left(\frac{3}{2}\pi+\theta\right)}-\frac{\sin\left(\frac{3}{2}\pi-\theta\right)}{\sin\left(\frac{\pi}{2}+\theta\right)\cos^2\theta}$$

$$=\frac{-\sin\theta(-\tan\theta)^2}{\sin\theta}-\frac{-\cos\theta}{\cos\theta\cos^2\theta}$$

$$=-\frac{\sin^2\theta}{\cos^2\theta}+\frac{1}{\cos^2\theta}$$

$$=\frac{-\sin^2\theta+1}{\cos^2\theta}$$

$$=\frac{\cos^2\theta}{\cos^2\theta}$$

$$=1$$

(4) $$\tan(\pi+\theta)\sin\left(\frac{\pi}{2}+\theta\right)+\frac{\cos(\pi-\theta)}{\tan(\pi-\theta)}$$

$$=\tan\theta\cos\theta+\frac{-\cos\theta}{-\tan\theta}$$

$$=\frac{\sin\theta}{\cos\theta}\cdot\cos\theta+\frac{\cos\theta}{\frac{\sin\theta}{\cos\theta}}$$

$$=\sin\theta+\frac{\cos^2\theta}{\sin\theta}$$

$$=\frac{\sin^2\theta+\cos^2\theta}{\sin\theta}$$

$$=\frac{1}{\sin\theta}$$

M 60 a Trigonometric Functions 6

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	1	2~3	4~5	6~

1. Convert the following angles from degrees to radians.

(1) $30° = \frac{\pi}{6}\ radians$

(2) $160° = \frac{8}{9}\pi\ radians$

(3) $-120° = -\frac{2}{3}\pi\ radians$

2. Convert the following angles from radians to degrees.

(1) $\frac{\pi}{3} radians = 60°$

(2) $\frac{7}{6}\pi\ radians = 210°$

(3) $-\frac{\pi}{4} radians = -45°$

3. Evaluate the following expressions.

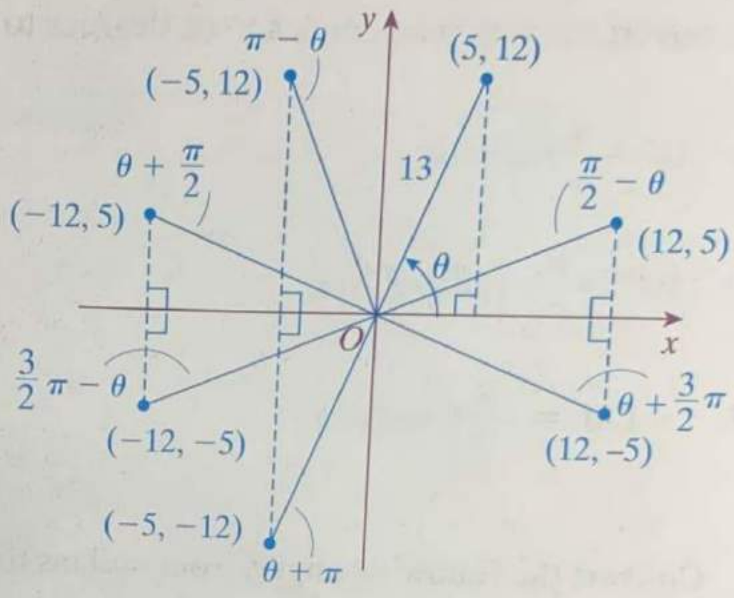
(1) $\sin\frac{\pi}{4} + \cos\frac{\pi}{6} = \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} = \frac{\sqrt{2}+\sqrt{3}}{2}$

(2) $\cos\frac{\pi}{3}\sin\frac{\pi}{6} = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

(3) $\tan\frac{\pi}{4}\tan\frac{\pi}{3} + \cos\frac{\pi}{3}\tan\frac{\pi}{6} = 1\cdot\sqrt{3} + \frac{1}{2}\cdot\frac{\sqrt{3}}{3} = \sqrt{3} + \frac{\sqrt{3}}{6} = \frac{7\sqrt{3}}{6}$

M 60 b

4. Given that $0<\theta<\frac{\pi}{2}$, and $\sin\theta=\frac{12}{13}$, draw the diagram, and use it to evaluate the following trigonometric expressions.



(1) $\sin\left(\theta+\frac{\pi}{2}\right)=\frac{5}{13}$

(2) $\sin\left(\theta+\frac{3}{2}\pi\right)=-\frac{5}{13}$

(3) $\sin\left(\frac{\pi}{2}-\theta\right)=\frac{5}{13}$

(4) $\sin(\pi-\theta)=\frac{12}{13}$

(5) $\cos\left(\frac{\pi}{2}-\theta\right)=\frac{12}{13}$

(6) $\cos(\pi-\theta)=-\frac{5}{13}$

(7) $\cos\left(\frac{3}{2}\pi-\theta\right)=-\frac{12}{13}$

(8) $\tan\left(\theta+\frac{\pi}{2}\right)=-\frac{5}{12}$

(9) $\tan(\pi+\theta)=\frac{12}{5}$

Trigonometric Functions 7

Time : to : Date Name

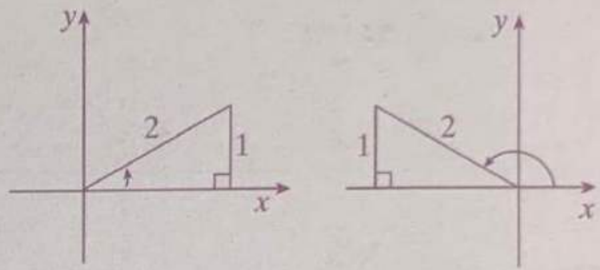
100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1~

Given that $0 \leq x < 2\pi$, solve each of the following equations in *radians*.

Ex. $\sin x = \dfrac{1}{2}$

From the graphs on the right,

$x = \dfrac{\pi}{6}, \dfrac{5}{6}\pi$



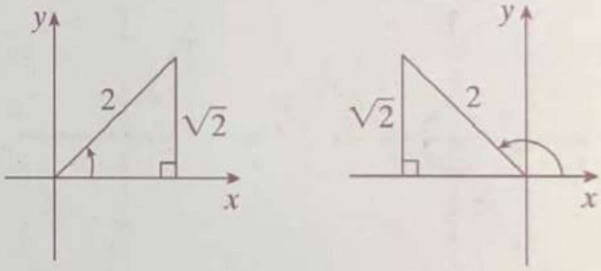
$x = \dfrac{\pi}{6}$ OR $x = \dfrac{5}{6}\pi$

Complete and check your answers.

$\sin x = \dfrac{\sqrt{2}}{2}$

From the graphs on the right,

$x = \boxed{\dfrac{\pi}{4}}, \boxed{\dfrac{3}{4}\pi}$



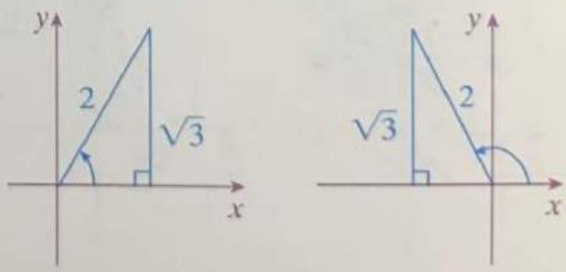
$x = \boxed{\dfrac{\pi}{4}}$ OR $x = \boxed{\dfrac{3}{4}\pi}$

Answers: $\dfrac{\pi}{4}, \dfrac{3}{4}\pi$

(1) $\sin x = \dfrac{\sqrt{3}}{2}$

From the graphs on the right,

$x = \dfrac{\pi}{3}, \dfrac{2}{3}\pi$



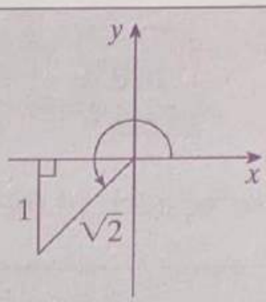
$x = \dfrac{\pi}{3}$ OR $x = \dfrac{2}{3}\pi$

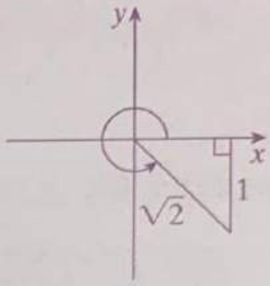
M 61 b

Ex. $\sin x = -\dfrac{1}{\sqrt{2}}$

From the graphs on the right,

$x = \dfrac{5}{4}\pi, \dfrac{7}{4}\pi$



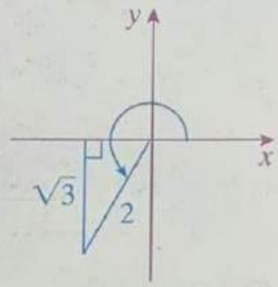


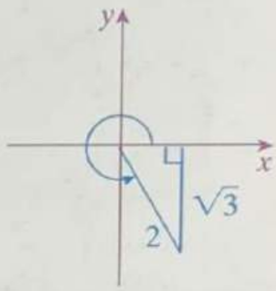
$x = \dfrac{5}{4}\pi$ OR $x = \dfrac{7}{4}\pi$

(2) $\sin x = -\dfrac{\sqrt{3}}{2}$

From the graphs on the right,

$x = \dfrac{4}{3}\pi, \dfrac{5}{3}\pi$



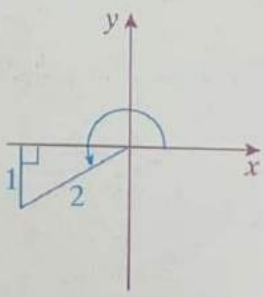


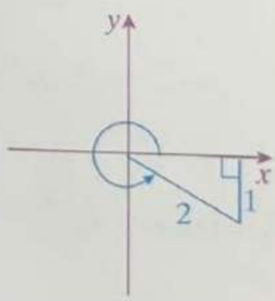
$x = \dfrac{4}{3}\pi$ OR $x = \dfrac{5}{3}\pi$

(3) $\sin x = -\dfrac{1}{2}$

From the graphs on the right,

$x = \dfrac{7}{6}\pi, \dfrac{11}{6}\pi$





$x = \dfrac{7}{6}\pi$ OR $x = \dfrac{11}{6}\pi$

M 62 a Trigonometric Functions 7

Time : to : Date Name

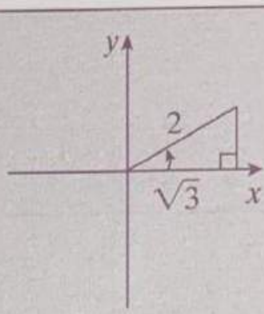
100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1~

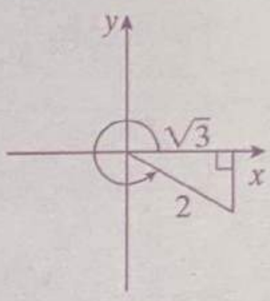
Given that $0 \leq x < 2\pi$, solve each of the following equations in *radians*.

Ex. $\cos x = \dfrac{\sqrt{3}}{2}$

From the graphs on the right,

$x = \dfrac{\pi}{6}, \dfrac{11}{6}\pi$





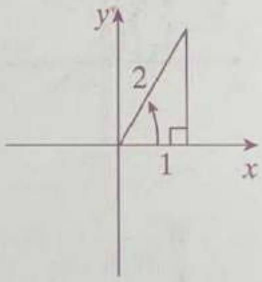
$x = \dfrac{\pi}{6}$ OR $x = \dfrac{11}{6}\pi$

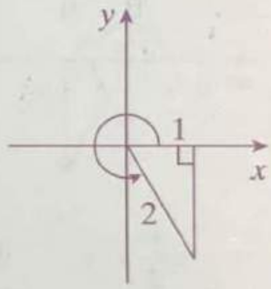
Complete and check your answers.

$\cos x = \dfrac{1}{2}$

From the graphs on the right,

$x = \boxed{\dfrac{\pi}{3}}, \boxed{\dfrac{5}{3}\pi}$





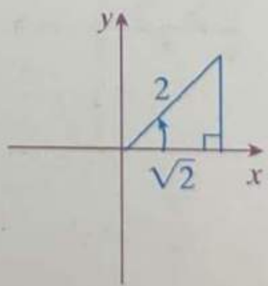
$x = \boxed{\dfrac{\pi}{3}}$ OR $x = \boxed{\dfrac{5}{3}\pi}$

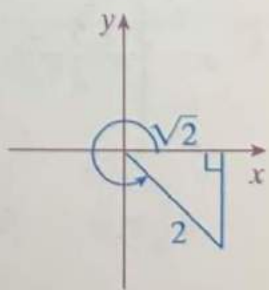
Answers. $\dfrac{\pi}{3}, \dfrac{5}{3}\pi$

(1) $\cos x = \dfrac{\sqrt{2}}{2}$

From the graphs on the right,

$x = \dfrac{\pi}{4}, \dfrac{7}{4}\pi$





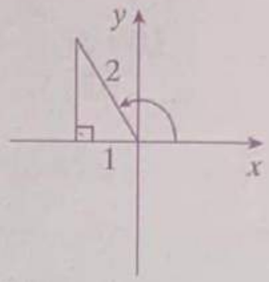
$x = \dfrac{\pi}{4}$ OR $x = \dfrac{7}{4}\pi$

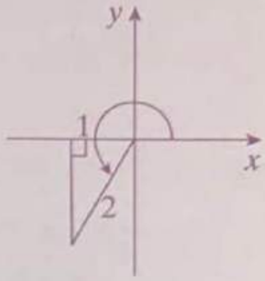
M 62 b

Ex. $\cos x = -\frac{1}{2}$

From the graphs on the right,

$x = \frac{2}{3}\pi, \frac{4}{3}\pi$



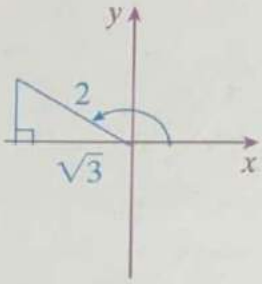


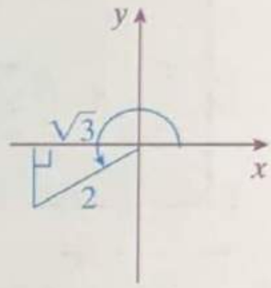
$x = \frac{2}{3}\pi$ OR $x = \frac{4}{3}\pi$

(2) $\cos x = -\frac{\sqrt{3}}{2}$

From the graphs on the right,

$x = \frac{5}{6}\pi, \frac{7}{6}\pi$



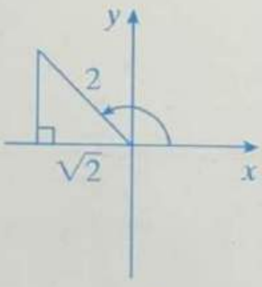


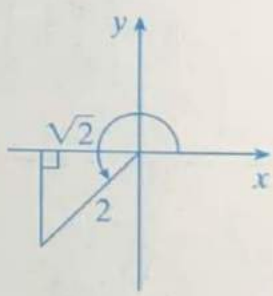
$x = \frac{5}{6}\pi$ OR $x = \frac{7}{6}\pi$

(3) $\cos x = -\frac{\sqrt{2}}{2}$

From the graphs on the right,

$x = \frac{3}{4}\pi, \frac{5}{4}\pi$





$x = \frac{3}{4}\pi$ OR $x = \frac{5}{4}\pi$

KUMON

M 63 a

Trigonometric Functions 7

Time : to : Date Name

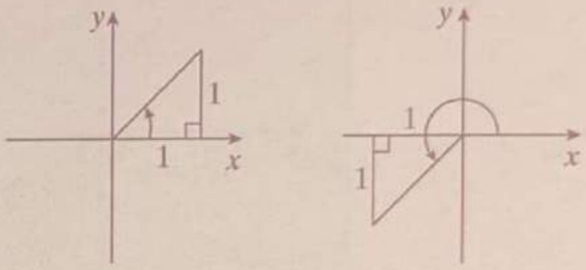
100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Given that $0 \leq x < 2\pi$, solve each of the following equations in *radians*.

Ex. $\tan x = 1$

From the graphs on the right,

$x = \frac{\pi}{4}, \frac{5}{4}\pi$

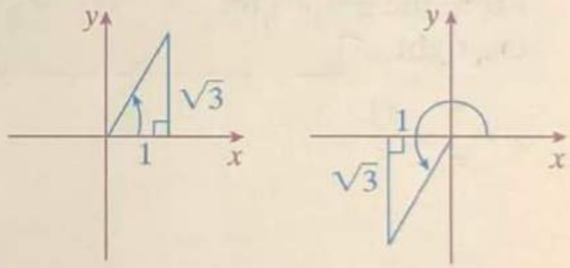


$x = \frac{\pi}{4}$ OR $x = \frac{5}{4}\pi$

(1) $\tan x = \sqrt{3}$

From the graphs on the right,

$x = \frac{\pi}{3}, \frac{4}{3}\pi$

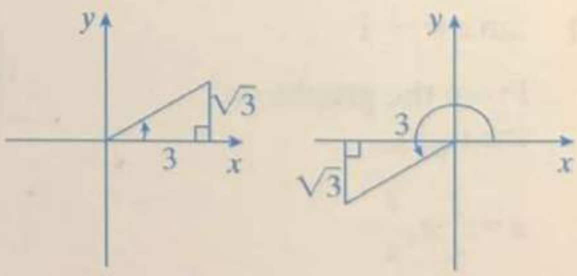


$x = \frac{\pi}{3}$ OR $x = \frac{4}{3}\pi$

(2) $\tan x = \frac{\sqrt{3}}{3}$

From the graphs on the right,

$x = \frac{\pi}{6}, \frac{7}{6}\pi$



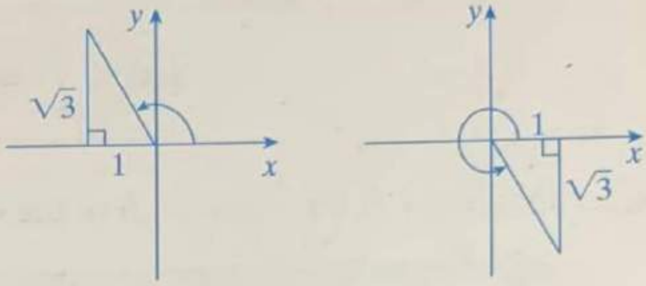
$x = \frac{\pi}{6}$ OR $x = \frac{7}{6}\pi$

M 63 b

(3) $\tan x = -\sqrt{3}$

From the graphs on the right,

$x = \frac{2}{3}\pi, \frac{5}{3}\pi$

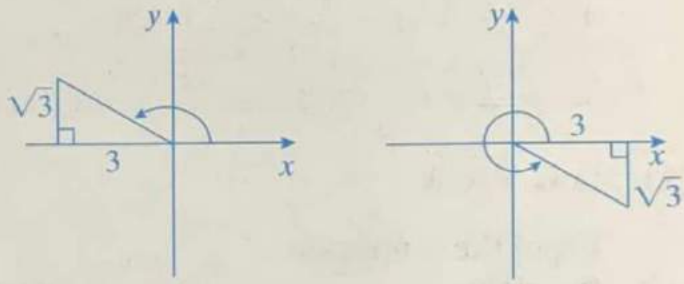


$x = \frac{2}{3}\pi$ OR $x = \frac{5}{3}\pi$

(4) $\tan x = -\frac{\sqrt{3}}{3}$

From the graphs on the right,

$x = \frac{5}{6}\pi, \frac{11}{6}\pi$

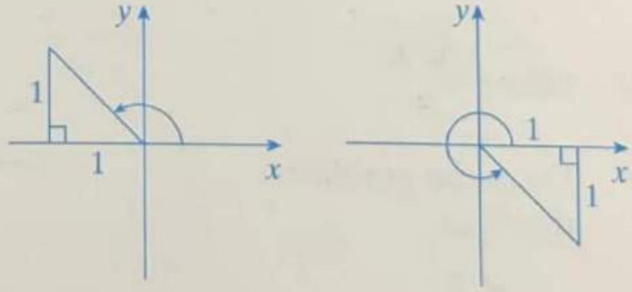


$x = \frac{5}{6}\pi$ OR $x = \frac{11}{6}\pi$

(5) $\tan x = -1$

From the graphs on the right,

$x = \frac{3}{4}\pi, \frac{7}{4}\pi$



$x = \frac{3}{4}\pi$ OR $x = \frac{7}{4}\pi$

M 64 a — Trigonometric Functions 7

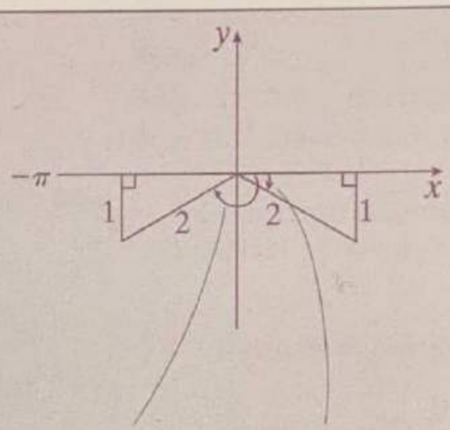
Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Given that $-\pi \leq x < \pi$, solve each of the following equations in *radians*.

Ex. $\sin x = -\frac{1}{2}$

$x = -\frac{5}{6}\pi, -\frac{\pi}{6}$



$x = -\frac{5}{6}\pi$ or $x = -\frac{\pi}{6}$

(1) $\cos x = -\frac{1}{\sqrt{2}}$

$x = -\frac{3}{4}\pi, \frac{3}{4}\pi$

(2) $\tan x = -\sqrt{3}$

$x = -\frac{\pi}{3}, \frac{2}{3}\pi$

(3) $\tan x = \frac{1}{\sqrt{3}}$

$x = -\frac{5}{6}\pi, \frac{\pi}{6}$

M 64 b

(4) $\tan x + \dfrac{1}{\sqrt{3}} = 0$

$$\tan x = -\frac{1}{\sqrt{3}}$$

$$x = -\frac{\pi}{6}, \frac{5}{6}\pi$$

(5) $2\cos x + \sqrt{3} = 0$

$$\cos x = -\frac{\sqrt{3}}{2}$$

$$x = -\frac{5}{6}\pi, \frac{5}{6}\pi$$

(6) $2\cos x + 1 = 0$

$$\cos x = -\frac{1}{2}$$

$$x = -\frac{2}{3}\pi, \frac{2}{3}\pi$$

M 65 a

Trigonometric Functions 7

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Given that $0 \leq x < 2\pi$, solve each of the following equations in *radians*.

Ex. $\sin 2x = \dfrac{\sqrt{3}}{2}$

Note:
The domain is in terms of x, whereas the trigonometric function is in terms of $2x$.

[Sol] Since $0 \leq x < 2\pi$,

$0 \leq 2x < 4\pi$

To convert it in terms of $2x$, multiply the given domain by 2.

$$2x = \frac{\pi}{3}, \frac{2}{3}\pi, \frac{7}{3}\pi, \frac{8}{3}\pi$$

Treat $2x$ as a single unit. $\left(\text{i.e., let } \alpha = 2x, \text{ where } 0 \leq \alpha < 4\pi \text{ and } \sin\alpha = \dfrac{\sqrt{3}}{2}\right)$

$$x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7}{6}\pi, \frac{4}{3}\pi$$

(1) $\cos 2x = \dfrac{\sqrt{2}}{2}$

[Sol] Since $0 \leq x < 2\pi$, $0 \leq 2x < 4\pi$

$$2x = \frac{\pi}{4}, \frac{7}{4}\pi, \frac{9}{4}\pi, \frac{15}{4}\pi$$

$$x = \frac{\pi}{8}, \frac{7}{8}\pi, \frac{9}{8}\pi, \frac{15}{8}\pi$$

(2) $\tan 2x = \dfrac{\sqrt{3}}{3}$

[Sol] Since $0 \leq x < 2\pi$, $0 \leq 2x < 4\pi$

$$2x = \frac{\pi}{6}, \frac{7}{6}\pi, \frac{13}{6}\pi, \frac{19}{6}\pi$$

$$x = \frac{\pi}{12}, \frac{7}{12}\pi, \frac{13}{12}\pi, \frac{19}{12}\pi$$

M 65 b

(3) $\sin\frac{x}{2}=\frac{1}{2}$

[Sol] Since $0\leq x<2\pi,\ 0\leq\frac{x}{2}<\boxed{\pi}$

$$\frac{x}{2}=\frac{\pi}{6},\frac{5}{6}\pi$$

$$x=\frac{\pi}{3},\frac{5}{3}\pi$$

(4) $2\cos\frac{x}{2}=\sqrt{2}$

[Sol] Since $0\leq x<2\pi,\ 0\leq\frac{x}{2}<\pi$

From $\cos\frac{x}{2}=\frac{\sqrt{2}}{2}$,

$$\frac{x}{2}=\frac{\pi}{4}$$

$$x=\frac{\pi}{2}$$

(5) $\tan 3x+1=0$

[Sol] Since $0\leq x<2\pi$,

$$0\leq 3x<6\pi$$

From $\tan 3x=-1$,

$$3x=\frac{3}{4}\pi,\frac{7}{4}\pi,\frac{11}{4}\pi,\frac{15}{4}\pi,\frac{19}{4}\pi,\frac{23}{4}\pi$$

$$x=\frac{\pi}{4},\frac{7}{12}\pi,\frac{11}{12}\pi,\frac{5}{4}\pi,\frac{19}{12}\pi,\frac{23}{12}\pi$$

Trigonometric Functions 7

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Given that $0 \leq x < 2\pi$, solve each of the following equations in *radians*.

Ex. $\sin\left(x+\frac{\pi}{3}\right)=\frac{1}{2}$

[Sol] Since $0 \leq x < 2\pi$,

$\frac{\pi}{3} \leq x+\frac{\pi}{3} < \frac{7}{3}\pi$

To convert the domain in terms of $\left(x+\frac{\pi}{3}\right)$, add $\frac{\pi}{3}$ to each part of the given domain.

$x+\frac{\pi}{3}=\frac{5}{6}\pi, \frac{13}{6}\pi$ ⟵ Treat $\left(x+\frac{\pi}{3}\right)$ as a single unit. $\left(\text{i.e., let } \alpha = x+\frac{\pi}{3}, \text{ where } \frac{\pi}{3} \leq \alpha < \frac{7}{3}\pi \text{ and } \sin\alpha=\frac{1}{2}\right)$

$x=\frac{\pi}{2}, \frac{11}{6}\pi$

(1) $\sin\left(x+\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$

[Sol] Since $0 \leq x < 2\pi$, $\frac{\pi}{6} \leq x+\frac{\pi}{6} < \frac{13}{6}\pi$

$x+\frac{\pi}{6}=\frac{\pi}{3}, \frac{2}{3}\pi$

$x=\frac{\pi}{6}, \frac{\pi}{2}$

(2) $\cos\left(x+\frac{\pi}{3}\right)=-\frac{1}{2}$

[Sol] Since $0 \leq x < 2\pi$, $\frac{\pi}{3} \leq x+\frac{\pi}{3} < \frac{7}{3}\pi$

$x+\frac{\pi}{3}=\frac{2}{3}\pi, \frac{4}{3}\pi$

$x=\frac{\pi}{3}, \pi$

M 66 b

(3) $\sin\left(x+\frac{\pi}{6}\right)=-\frac{\sqrt{3}}{2}$

[Sol] Since $0\le x<2\pi$, $\frac{\pi}{6}\le x+\frac{\pi}{6}<\frac{13}{6}\pi$

$$x+\frac{\pi}{6}=\frac{4}{3}\pi,\ \frac{5}{3}\pi$$

$$x=\frac{7}{6}\pi,\ \frac{3}{2}\pi$$

(4) $\cos\left(x+\frac{\pi}{6}\right)=\frac{\sqrt{2}}{2}$

[Sol] Since $0\le x<2\pi$, $\frac{\pi}{6}\le x+\frac{\pi}{6}<\frac{13}{6}\pi$

$$x+\frac{\pi}{6}=\frac{\pi}{4},\ \frac{7}{4}\pi$$

$$x=\frac{\pi}{12},\ \frac{19}{12}\pi$$

(5) $\tan\left(x+\frac{\pi}{3}\right)=-\sqrt{3}$

[Sol] Since $0\le x<2\pi$, $\frac{\pi}{3}\le x+\frac{\pi}{3}<\frac{7}{3}\pi$

$$x+\frac{\pi}{3}=\frac{2}{3}\pi,\ \frac{5}{3}\pi$$

$$x=\frac{\pi}{3},\ \frac{4}{3}\pi$$

Trigonometric Functions 7

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Given that $-\pi < x < \pi$, solve each of the following equations in *radians*.

(1) $\sin 2x = \frac{1}{2}$

[Sol] Solve over domain: $\boxed{-2\pi} < 2x < \boxed{2\pi}$

$$2x = -\frac{11}{6}\pi,\ -\frac{7}{6}\pi,\ \frac{\pi}{6},\ \frac{5}{6}\pi$$

$$x = -\frac{11}{12}\pi,\ -\frac{7}{12}\pi,\ \frac{\pi}{12},\ \frac{5}{12}\pi$$

(2) $\cos 2x = -\frac{\sqrt{3}}{2}$

[Sol] Solve over domain: $-2\pi < 2x < 2\pi$

$$2x = -\frac{7}{6}\pi,\ -\frac{5}{6}\pi,\ \frac{5}{6}\pi,\ \frac{7}{6}\pi$$

$$x = -\frac{7}{12}\pi,\ -\frac{5}{12}\pi,\ \frac{5}{12}\pi,\ \frac{7}{12}\pi$$

(3) $3\tan\frac{x}{3} - \sqrt{3} = 0$

[Sol] Solve $\tan\frac{x}{3} = \frac{\sqrt{3}}{3}$ over domain: $-\frac{\pi}{3} < \frac{x}{3} < \frac{\pi}{3}$

$$\frac{x}{3} = \frac{\pi}{6}$$

$$x = \frac{\pi}{2}$$

M 67 b

(4) $\tan\left(2x-\frac{\pi}{4}\right)=1$

[Sol] Solve $\tan\left(2x-\frac{\pi}{4}\right)=1$ over domain: $-\frac{9}{4}\pi<2x-\frac{\pi}{4}<\frac{7}{4}\pi$

$$2x-\frac{\pi}{4}=-\frac{7}{4}\pi,\ -\frac{3}{4}\pi,\ \frac{\pi}{4},\ \frac{5}{4}\pi$$

$$x=-\frac{3}{4}\pi,\ -\frac{\pi}{4},\ \frac{\pi}{4},\ \frac{3}{4}\pi$$

(5) $\sin\left(\frac{x}{2}-\frac{\pi}{6}\right)=-\frac{\sqrt{3}}{2}$

[Sol] Solve $\sin\left(\frac{x}{2}-\frac{\pi}{6}\right)=-\frac{\sqrt{3}}{2}$ over domain: $-\frac{2}{3}\pi<\frac{x}{2}-\frac{\pi}{6}<\frac{\pi}{3}$

$$\frac{x}{2}-\frac{\pi}{6}=-\frac{\pi}{3}$$

$$x=-\frac{\pi}{3}$$

(6) $\sqrt{2}\cos\left(\frac{x}{3}+\frac{\pi}{2}\right)-1=0$

[Sol] Solve $\cos\left(\frac{x}{3}+\frac{\pi}{2}\right)=\frac{1}{\sqrt{2}}$ over domain: $\frac{\pi}{6}<\frac{x}{3}+\frac{\pi}{2}<\frac{5}{6}\pi$

$$\frac{x}{3}+\frac{\pi}{2}=\frac{\pi}{4}$$

$$x=-\frac{3}{4}\pi$$

M 68 a

Trigonometric Functions 7

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Given that $0 \leq x < 2\pi$, solve each of the following equations in *radians*.

Ex. $\sqrt{3}\sin x - 2\sin^2 x = 0$

[Sol] $\sin x(\sqrt{3} - 2\sin x) = 0$

$\sin x = 0, \quad \frac{\sqrt{3}}{2}$

$x = 0, \quad \frac{\pi}{3}, \quad \frac{2}{3}\pi, \quad \pi$

(1) $\sqrt{2}\cos x - 2\cos^2 x = 0$

[Sol] $\cos x(\sqrt{2} - 2\cos x) = 0$

$\cos x = 0, \quad \frac{\sqrt{2}}{2}$

$x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3}{2}\pi, \frac{7}{4}\pi$

(2) $2\cos^2 x - 3\cos x + 1 = 0$

[Sol] $(2\cos x - 1)(\cos x - 1) = 0$

$\cos x = \frac{1}{2}, \quad 1$

$x = 0, \frac{\pi}{3}, \frac{5}{3}\pi$

M 68 b

(3) $2\sin^2 x + 3\sin x - 2 = 0$ **Hint**

[Sol] $(2\sin x - 1)(\sin x + 2) = 0$

$\sin x = \dfrac{1}{2} \quad (\because \sin x + 2 \neq 0)$

Since $\sin x + 2 \neq 0$, we only need to examine the case $2\sin x - 1 = 0$.

$x = \dfrac{\pi}{6}, \ \dfrac{5}{6}\pi$

(4) $3\sin x = 2\cos^2 x$ (Let $\cos^2 x = 1 - \sin^2 x$)

[Sol] $3\sin x = 2(1 - \sin^2 x)$

$2\sin^2 x + 3\sin x - 2 = 0$

$(2\sin x - 1)(\sin x + 2) = 0$

$\sin x = \dfrac{1}{2} \quad (\because \sin x + 2 \neq 0)$

$x = \dfrac{\pi}{6}, \ \dfrac{5}{6}\pi$

(5) $2\cos^2 x + 10\cos x = 5\sqrt{3} + \sqrt{3}\cos x$

[Sol] $2\cos^2 x + 10\cos x - \sqrt{3}\cos x - 5\sqrt{3} = 0$

$2\cos^2 x + (10 - \sqrt{3})\cos x - 5\sqrt{3} = 0$

$(2\cos x - \sqrt{3})(\cos x + 5) = 0$

$\cos x = \dfrac{\sqrt{3}}{2} \quad (\because \cos x + 5 \neq 0)$

$x = \dfrac{\pi}{6}, \ \dfrac{11}{6}\pi$

Hint

$\sin x + 2 \neq 0$

KUMON

M 69 a Trigonometric Functions 7

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. Given that $0 \leq x < 2\pi$, solve each of the following equations in *radians*.

(1) $2\sin^2 x + \cos x = 1$

[Sol] $2(1 - \cos^2 x) + \cos x - 1 = 0$

$2\cos^2 x - \cos x - 1 = 0$

$(2\cos x + 1)(\cos x - 1) = 0$

$\cos x = -\frac{1}{2},\quad 1$

$x = 0,\quad \frac{2}{3}\pi,\quad \frac{4}{3}\pi$

(2) $2(\cos^2 x + 1) = 5\cos x$

[Sol] $2\cos^2 x - 5\cos x + 2 = 0$

$(2\cos x - 1)(\cos x - 2) = 0$

$\cos x = \frac{1}{2} \qquad (\because \cos x - 2 \neq 0)$

$x = \frac{\pi}{3},\quad \frac{5}{3}\pi$

(3) $3(\tan^2 x - 1) = 2\sqrt{3}\tan x$

[Sol] $3\tan^2 x - 2\sqrt{3}\tan x - 3 = 0$

$(3\tan x + \sqrt{3})(\tan x - \sqrt{3}) = 0$ [or: $(\sqrt{3}\tan x - 3)(\sqrt{3}\tan x + 1) = 0$]

$\tan x = -\frac{\sqrt{3}}{3},\quad \sqrt{3}$

$x = \frac{\pi}{3},\quad \frac{5}{6}\pi,\quad \frac{4}{3}\pi,\quad \frac{11}{6}\pi$

M 69 b

2. Given that $-\pi < x < \pi$, solve each of the following equations in *radians*.

(1) $4\sin^2 x - 2(\sqrt{3}+1)\sin x + \sqrt{3} = 0$

[Sol] $(2\sin x - 1)(2\sin x - \sqrt{3}) = 0$

$$\sin x = \frac{1}{2},\ \frac{\sqrt{3}}{2}$$

$$x = \frac{\pi}{6},\ \frac{\pi}{3},\ \frac{2}{3}\pi,\ \frac{5}{6}\pi$$

(2) $\sqrt{3}\tan^2 x - (\sqrt{3}+1)\tan x + 1 = 0$

[Sol] $(\sqrt{3}\tan x - 1)(\tan x - 1) = 0$

$$\tan x = \frac{1}{\sqrt{3}},\ 1$$

$$x = -\frac{5}{6}\pi,\ -\frac{3}{4}\pi,\ \frac{\pi}{6},\ \frac{\pi}{4}$$

(3) $3\tan^2 x - \sqrt{3} = (3-\sqrt{3})\tan x$

[Sol] $3\tan^2 x - (3-\sqrt{3})\tan x - \sqrt{3} = 0$

$(3\tan x + \sqrt{3})(\tan x - 1) = 0$

$$\tan x = -\frac{\sqrt{3}}{3},\ 1$$

$$x = -\frac{3}{4}\pi,\ -\frac{\pi}{6},\ \frac{\pi}{4},\ \frac{5}{6}\pi$$

KUMON

M 70 a Trigonometric Functions 7

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. Given that $0 \leq x < 2\pi$, solve each of the following equations in *radians*.

(1) $\cos 2x - \dfrac{1}{2} = 0$

[Sol] $\cos 2x = \dfrac{1}{2}$

Solve over $0 \leq 2x < 4\pi$

$2x = \dfrac{\pi}{3}, \dfrac{5}{3}\pi, \dfrac{7}{3}\pi, \dfrac{11}{3}\pi$

$x = \dfrac{\pi}{6}, \dfrac{5}{6}\pi, \dfrac{7}{6}\pi, \dfrac{11}{6}\pi$

(2) $2\sin\dfrac{x}{2} = \sqrt{3}$

[Sol] $\sin\dfrac{x}{2} = \dfrac{\sqrt{3}}{2}$

Solve over $0 \leq \dfrac{x}{2} < \pi$

$\dfrac{x}{2} = \dfrac{\pi}{3}, \dfrac{2}{3}\pi$

$x = \dfrac{2}{3}\pi, \dfrac{4}{3}\pi$

(3) $3\tan\left(x + \dfrac{\pi}{6}\right) = \sqrt{3}$

[Sol] Solve over $\dfrac{\pi}{6} \leq x + \dfrac{\pi}{6} < \dfrac{13}{6}\pi$

From $\tan\left(x + \dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{3}$,

$x + \dfrac{\pi}{6} = \dfrac{\pi}{6}, \dfrac{7}{6}\pi$

$x = 0, \pi$

M 70 b

2. Given that $-\pi \leq x < \pi$, solve each of the following equations in *radians*.

(1) $\sin\left(x+\dfrac{\pi}{3}\right)=-\dfrac{1}{2}$

[Sol] Solve over $-\dfrac{2}{3}\pi \leq x+\dfrac{\pi}{3} < \dfrac{4}{3}\pi$

$$x+\frac{\pi}{3}=-\frac{\pi}{6},\ \frac{7}{6}\pi$$

$$x=-\frac{\pi}{2},\ \frac{5}{6}\pi$$

(2) $\cos\left(\dfrac{x}{2}-\dfrac{\pi}{6}\right)=-\dfrac{1}{2}$

[Sol] Solve over $-\dfrac{2}{3}\pi \leq \dfrac{x}{2}-\dfrac{\pi}{6} < \dfrac{\pi}{3}$

$$\frac{x}{2}-\frac{\pi}{6}=-\frac{2}{3}\pi$$

$$x=-\pi$$

(3) $2\sin^2 x-3\cos x=0$ (Let $\sin^2 x=1-\cos^2 x$)

[Sol] $2(1-\cos^2 x)-3\cos x=0$

$$2\cos^2 x+3\cos x-2=0$$

$$(2\cos x-1)(\cos x+2)=0$$

$$\cos x=\frac{1}{2} \quad (\because \cos x+2\neq 0)$$

$$x=-\frac{\pi}{3},\ \frac{\pi}{3}$$

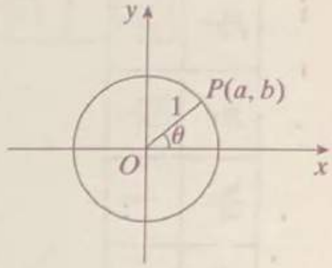
Graphs of Trigonometric Functions 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1~

Given a unit circle (a circle with radius 1) with center at the origin, letting $P(a, b)$ be a point on the circle, and letting θ be the shortest angle formed counterclockwise from the positive x-axis to OP,

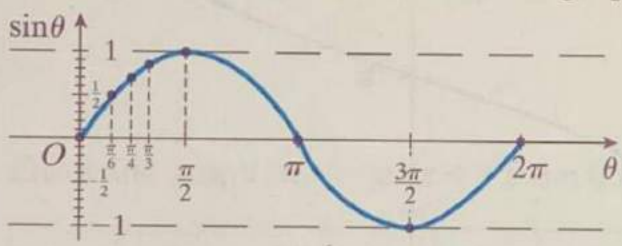
$$\sin\theta = \frac{b}{1} = \boxed{b}$$



Reviewing the values of $\sin\theta$ for some common angles:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0

Plotting the above values, trace the graph of $\sin\theta$ from 0 to 2π.

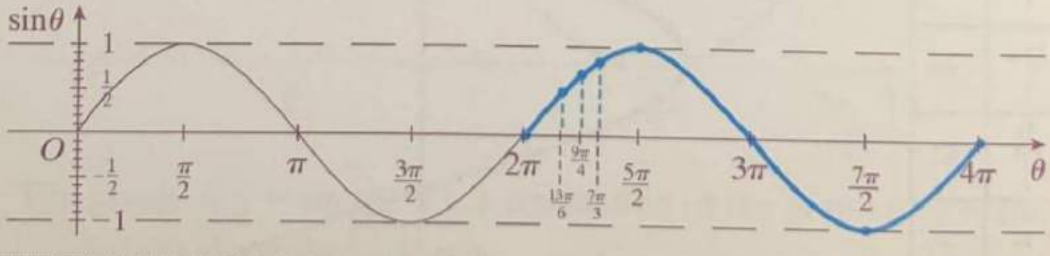


Answers: $b, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1, 0, -1, 0$

Complete and check your answers.

θ	2π	$\frac{13\pi}{6}$	$\frac{9\pi}{4}$	$\frac{7\pi}{3}$	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0

Continue to sketch the graph of $\sin\theta$ from 2π to 4π.

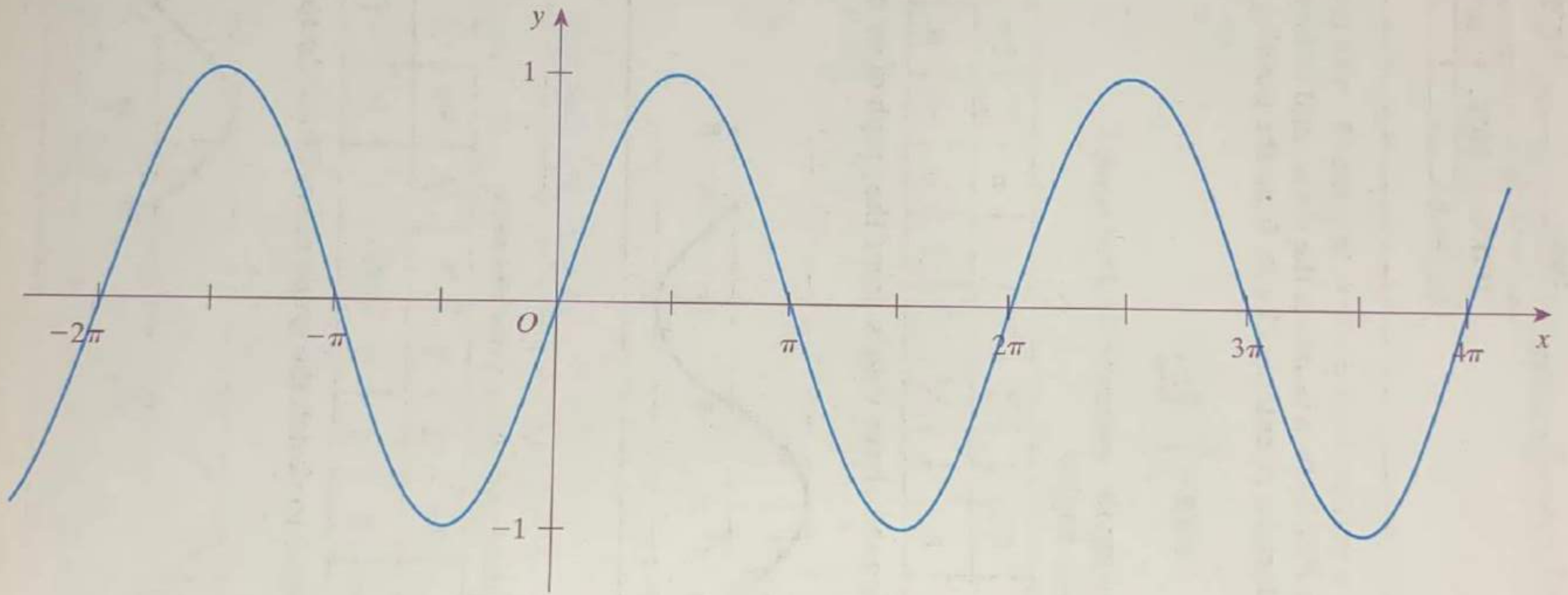


Answers: $0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1, 0, -1, 0$

M 71 b

1. Draw the graph of $y = \sin x$ from -2π to 4π.

x	-2π	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{13\pi}{6}$	$\frac{9\pi}{4}$	$\frac{7\pi}{3}$	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
y	0	1	0	-1	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0



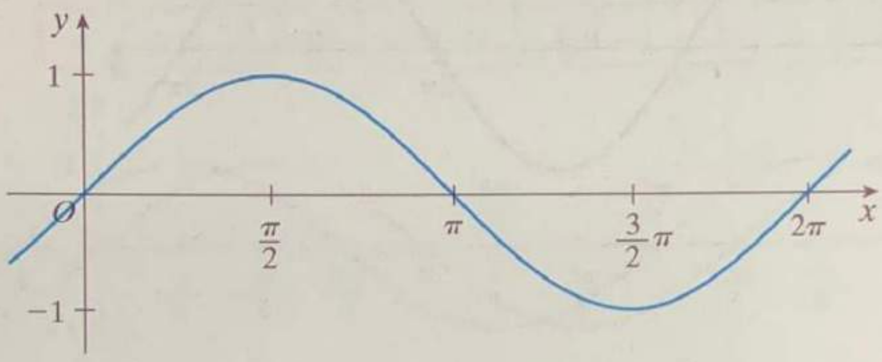
Graphs of Trigonometric Functions 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

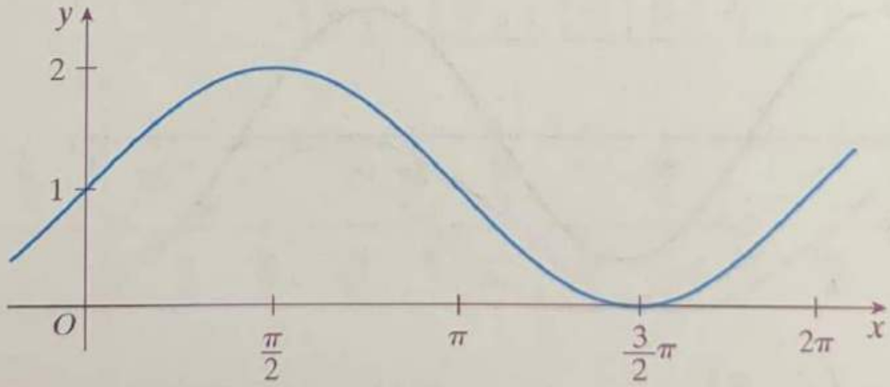
1. Draw the graph of $y = \sin x$ from 0 to 2π.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0



2. Draw the graph of $y = \sin x + 1$ from 0 to 2π.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	1	$\frac{3}{2}$	$\frac{2+\sqrt{2}}{2}$	$\frac{2+\sqrt{3}}{2}$	2	1	0	1

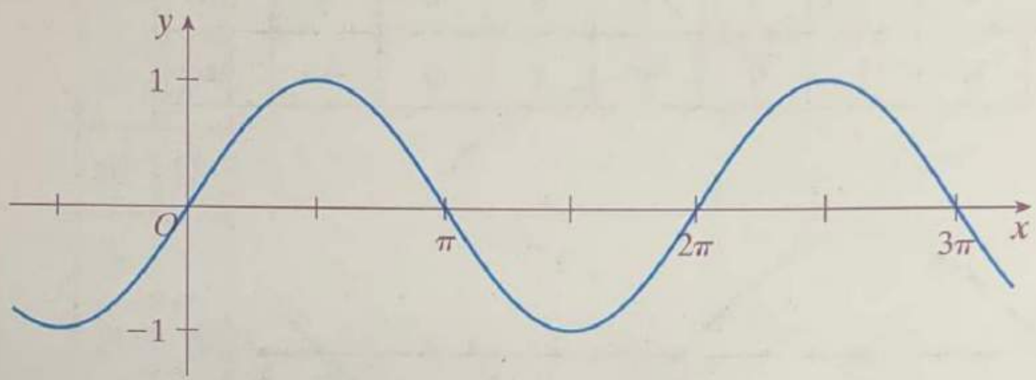


The graph of $y = \sin x + 1$ is a translation of the graph of $y = \sin x$, $\boxed{1}$ unit(s) along the $\boxed{y}$ -axis.

M 72 b

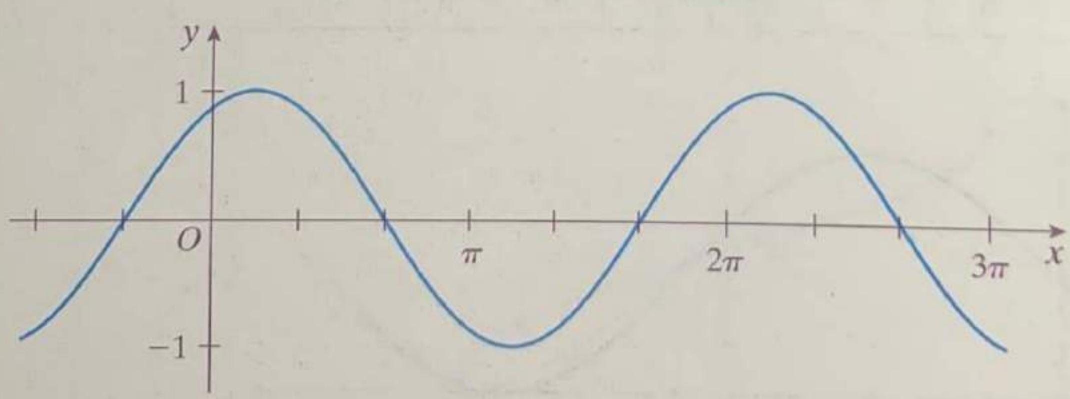
3. Draw the graph of $y = \sin x$ from $-\frac{\pi}{2}$ to 3π.

x	$-\frac{\pi}{2}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{13\pi}{6}$	$\frac{9\pi}{4}$	$\frac{7\pi}{3}$	$\frac{5\pi}{2}$	3π
y	-1	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0



4. Draw the graph of $y = \sin\left(x + \frac{\pi}{3}\right)$ from $-\frac{\pi}{2}$ to 3π.

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π	$\frac{13\pi}{6}$	$\frac{8\pi}{3}$	3π
y	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	0	$-\frac{\sqrt{3}}{2}$



The graph of $y = \sin\left(x + \frac{\pi}{3}\right)$ is a translation of the graph of $y = \sin x$,

$-\frac{\pi}{3}$ unit(s) along the $\boxed{x}$ -axis.

M 73 a Graphs of Trigonometric Functions 1

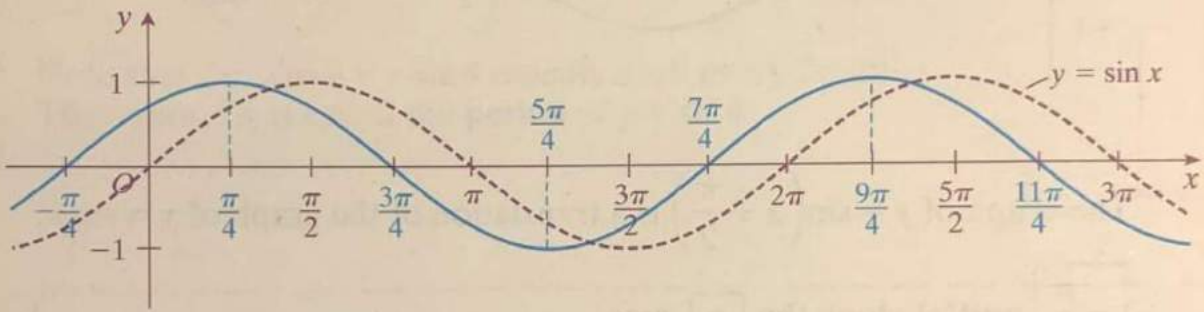
Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

1. Use the graph of $y=\sin x$ as a reference to draw the graph of each of the following trigonometric functions, and state the translation.

(1) $y=\sin\left(x+\frac{\pi}{4}\right)$

x	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$
y	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0

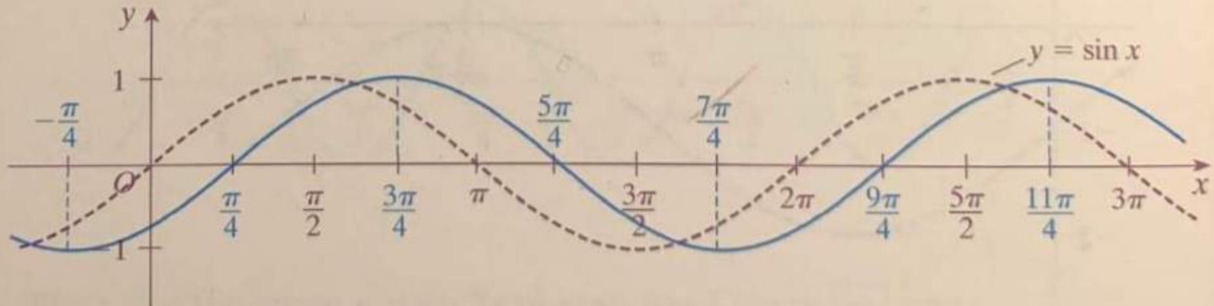


The graph of $y=\sin\left(x+\frac{\pi}{4}\right)$ is a translation of the graph of $y=\sin x$,

$\boxed{-\frac{\pi}{4}}$ unit(s) along the $\boxed{x}$ -axis.

(2) $y=\sin\left(x-\frac{\pi}{4}\right)$

x	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$

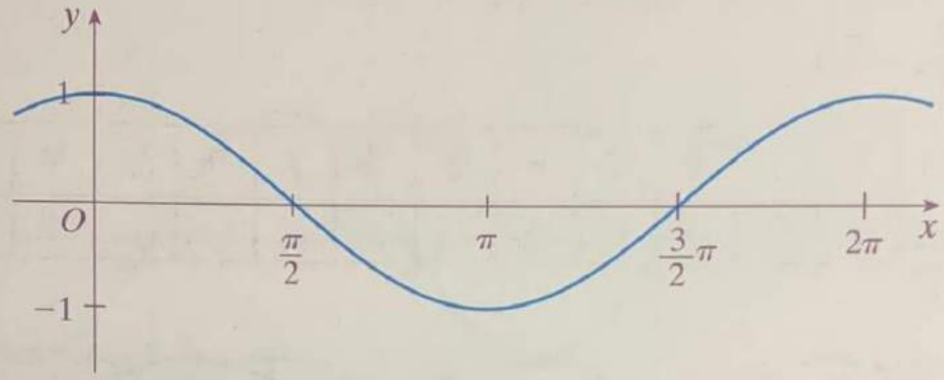


The graph of $y=\sin\left(x-\frac{\pi}{4}\right)$ is a translation of the graph of $y=\sin x$,

$\boxed{\frac{\pi}{4}}$ unit(s) along the $\boxed{x}$ -axis.

2. Draw the graph of each of the following trigonometric functions, and state the translation.

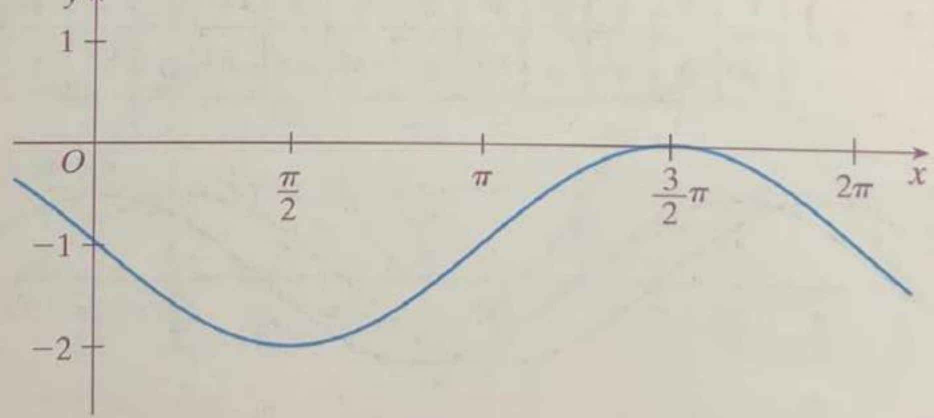
(1) $y = \sin\left(x + \dfrac{\pi}{2}\right)$



The graph of $y = \sin\left(x + \dfrac{\pi}{2}\right)$ is a translation of the graph of $y = \sin x$,

$\boxed{-\dfrac{\pi}{2}}$ unit(s) along the $\boxed{x}$ -axis.

(2) $y = \sin(x - \pi) - 1$



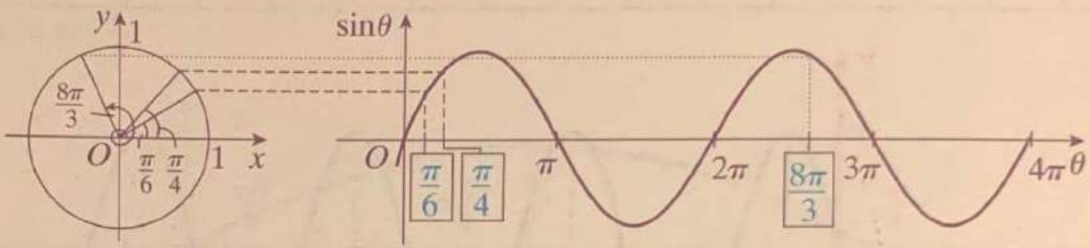
The graph of $y = \sin(x - \pi) - 1$ is a translation of the graph of $y = \sin x$, $\boxed{\pi}$ units along the x-axis and $\boxed{-1}$ unit(s) along the y-axis.

M 74 a Graphs of Trigonometric Functions 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1~

Notice below how the sine curve stems from the graph of the sine function between 0 and 2π on the unit circle.



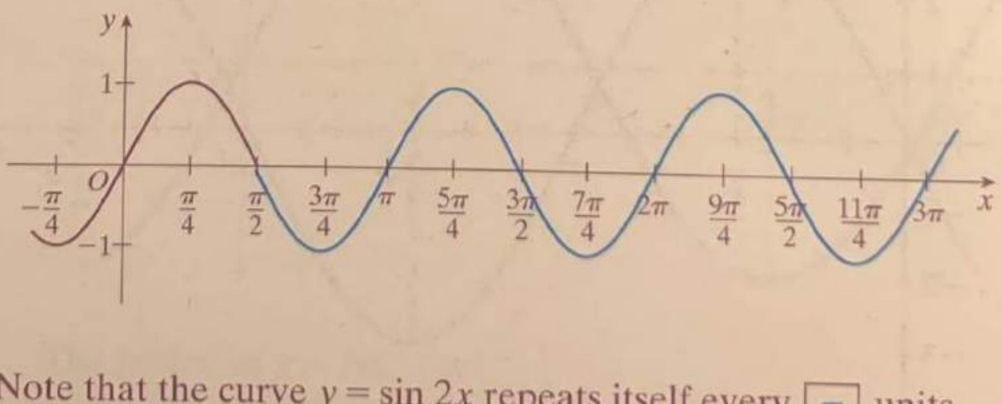
Note that the curve $y = \sin\theta$ repeats itself every 2π units.
Therefore, 2π is called the **period** of $y = \sin\theta$.

Answers: $\frac{\pi}{6}, \frac{\pi}{4}, \frac{8\pi}{3}$

Complete and check your answers.

Continue to sketch the graph of $y = \sin 2x$, and state the period.

x	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	-1	0	1	0	-1	0	1	0	-1	0	1	0	-1	0



Note that the curve $y = \sin 2x$ repeats itself every $\boxed{\pi}$ units.
Therefore, $\boxed{\pi}$ is the period of $y = \sin 2x$.

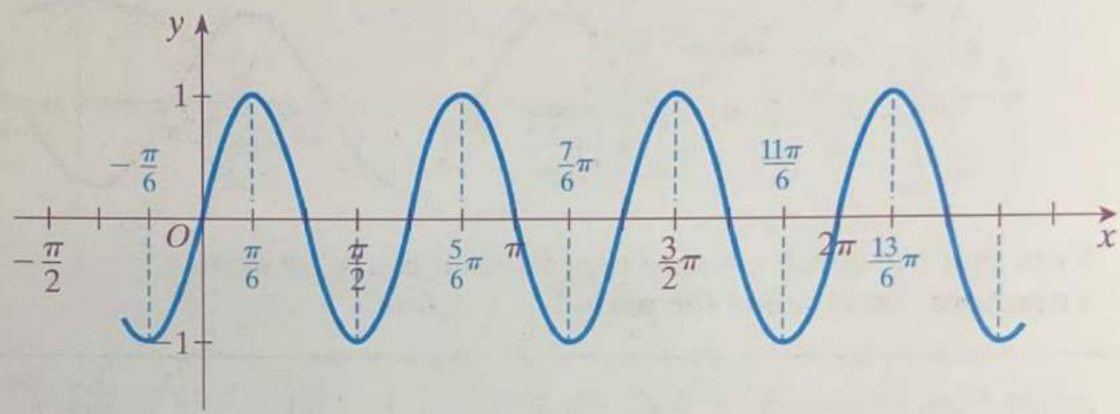
Answers: $-1, 0, 1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, 0$, π, π

M 74 b

Draw the graph of each of the following trigonometric functions and state the period.

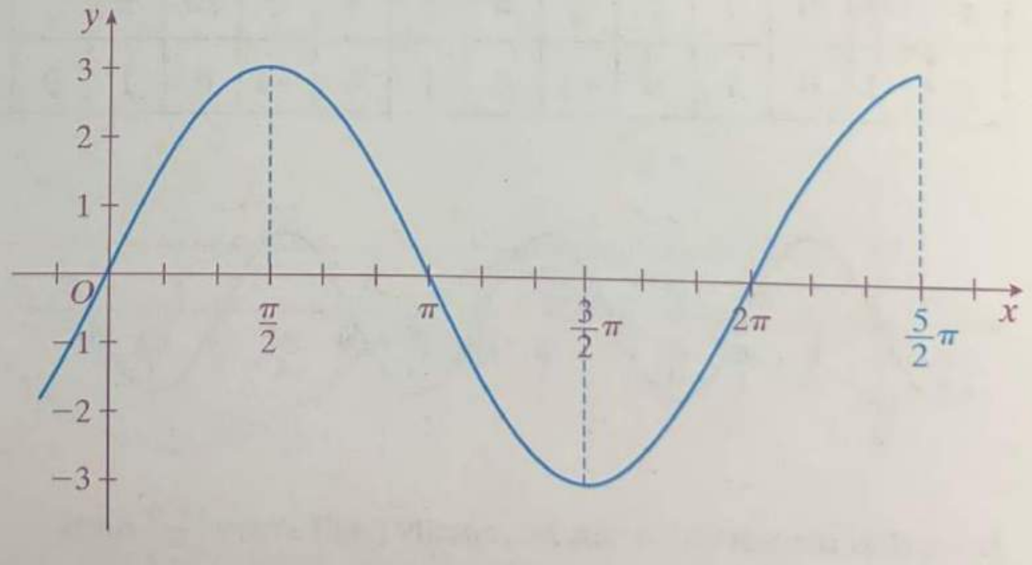
(1) $y = \sin 3x$

x	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π	$\frac{13\pi}{6}$	$\frac{7\pi}{3}$
y	-1	0	1	0	-1	0	1	0	-1	0	1	0	-1	0	1	0



The period of $y = \sin 3x$ is $\boxed{\frac{2}{3}\pi}$.

(2) $y = 3\sin x$



The period of $y = 3\sin x$ is $\boxed{2\pi}$.

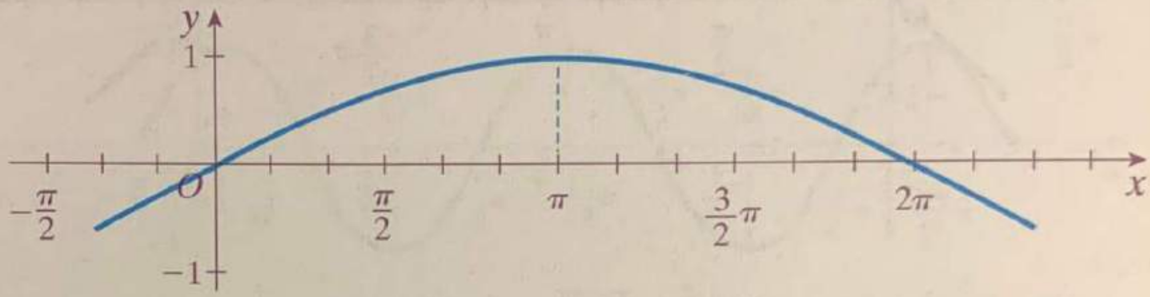
Graphs of Trigonometric Functions 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

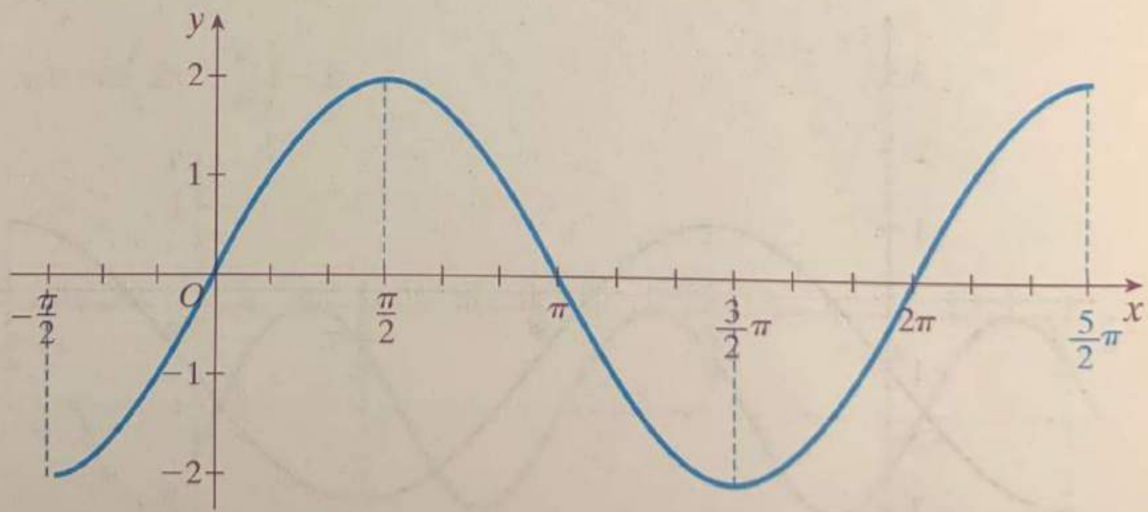
Draw the graph of each of the following trigonometric functions and state the period.

(1) $y=\sin\frac{x}{2}$



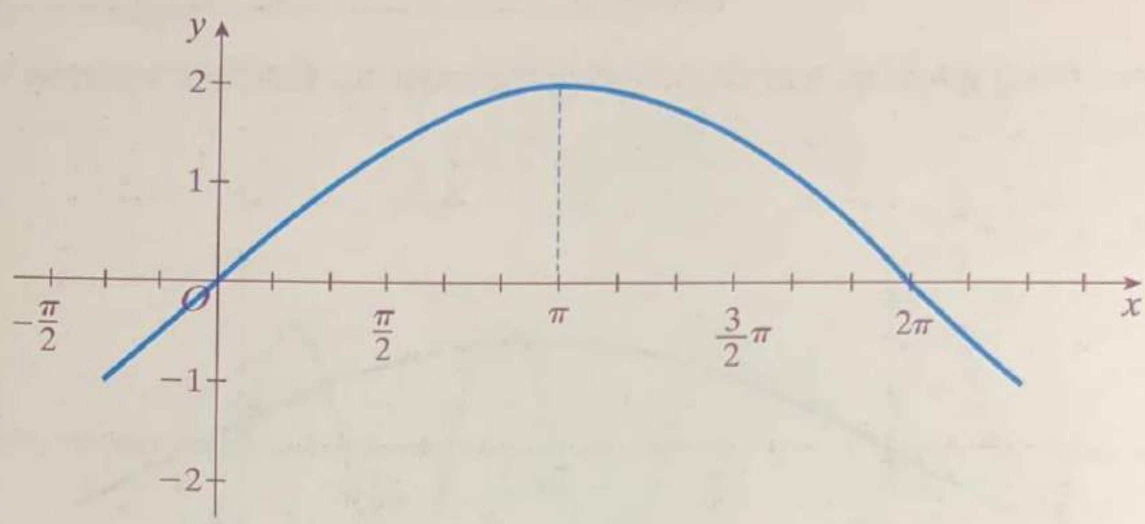
The period of $y=\sin\frac{x}{2}$ is $\boxed{4\pi}$.

(2) $y=2\sin x$



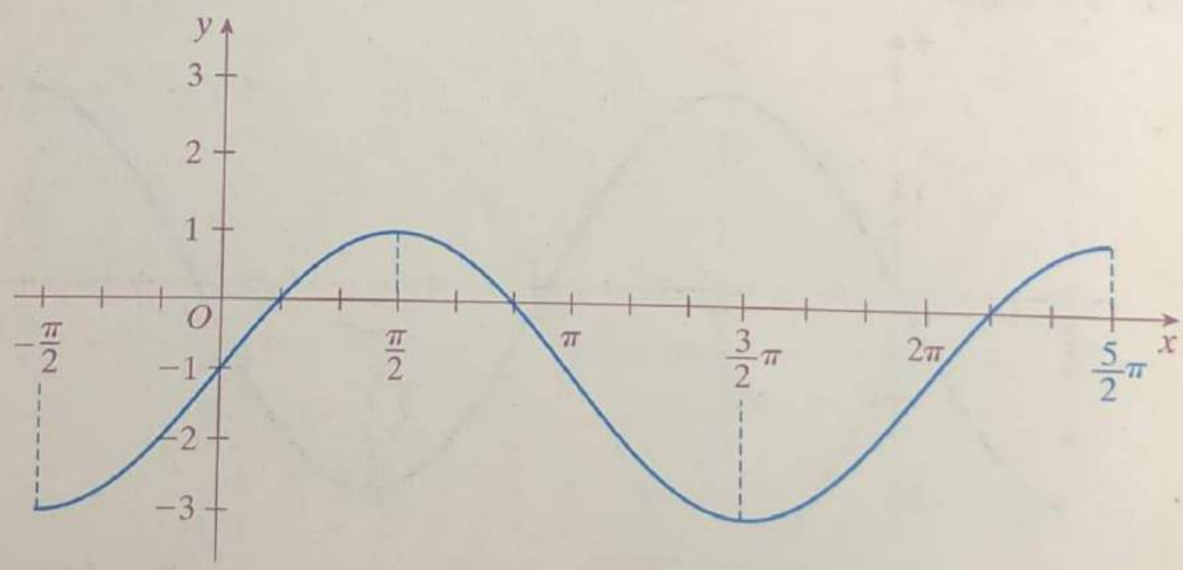
The period of $y=2\sin x$ is $\boxed{2\pi}$.

(3) $y=2\sin\dfrac{x}{2}$



The period of $y=2\sin\dfrac{x}{2}$ is $\boxed{4\pi}$.

(4) $y=2\sin x-1$



The period of $y=2\sin x-1$ is $\boxed{2\pi}$.

KUMON

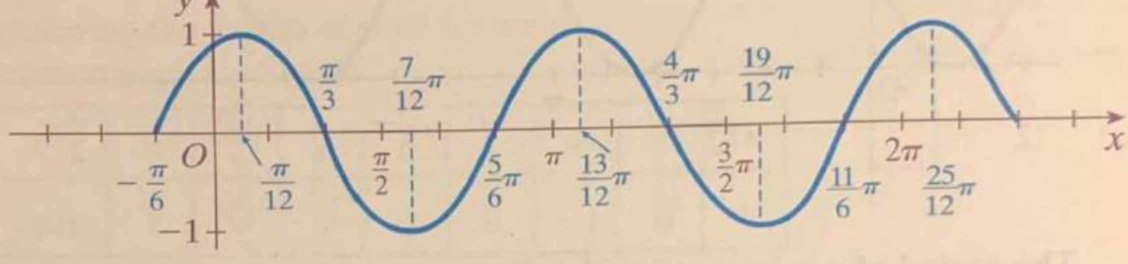
M 76 a Graphs of Trigonometric Functions 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

1. Draw the graph of each of the following trigonometric functions and state the translation.

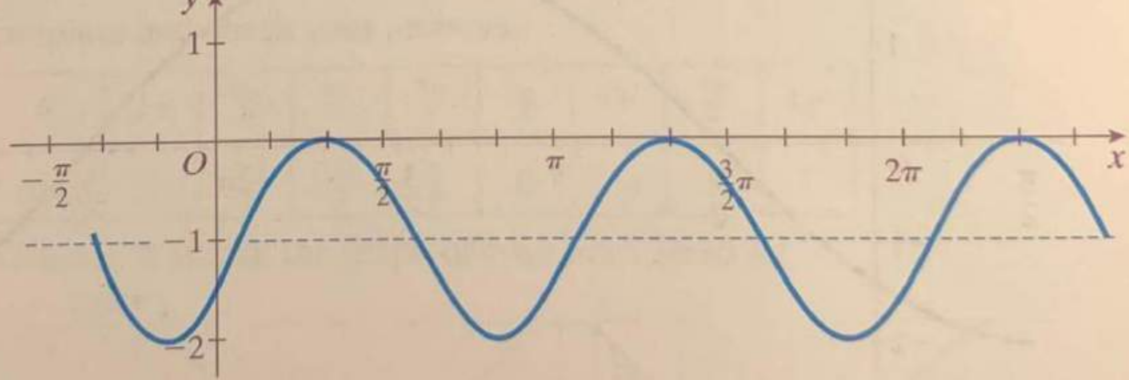
(1) $y=\sin\left(2x+\frac{\pi}{3}\right)$



Since $\sin\left(2x+\frac{\pi}{3}\right)=\sin 2\left(x+\boxed{\frac{\pi}{6}}\right)$,

The graph of $y=\sin\left(2x+\frac{\pi}{3}\right)$ is a translation of $y=\sin 2x$, $\boxed{-\frac{\pi}{6}}$ units along the x-axis.

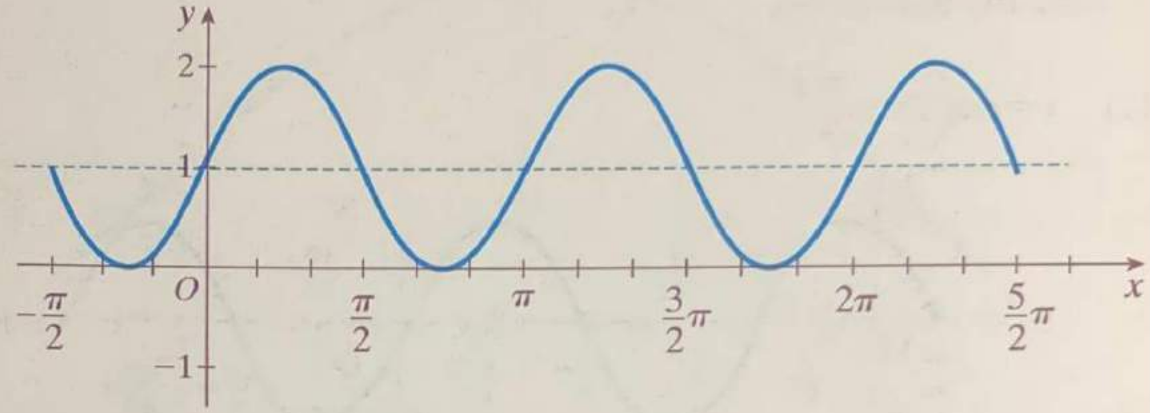
(2) $y=\sin\left(2x-\frac{\pi}{6}\right)-1$



The graph of $y=\sin\left(2x-\frac{\pi}{6}\right)-1$ is a translation of $y=\sin 2x$, $\boxed{\frac{\pi}{12}}$ units along the x-axis, and $\boxed{-1}$ unit(s) along the y-axis.

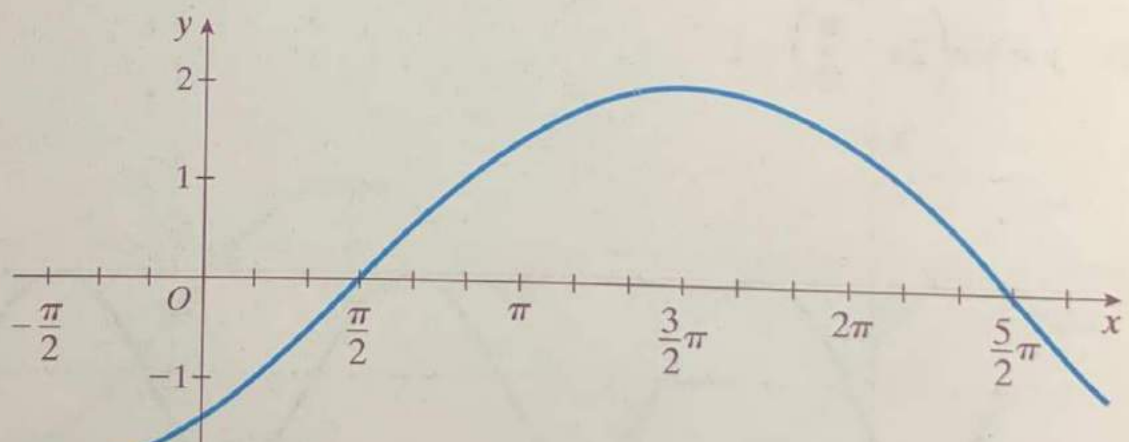
2. Draw the graph of each of the following trigonometric functions, and state the period.

(1) $y = 1 + \sin 2x$



The period of $y = 1 + \sin 2x$ is $\boxed{\pi}$.

(2) $y = 2\sin\frac{1}{2}\left(x - \frac{\pi}{2}\right)$



The period of $y = 2\sin\frac{1}{2}\left(x - \frac{\pi}{2}\right)$ is $\boxed{4\pi}$.

KUMON

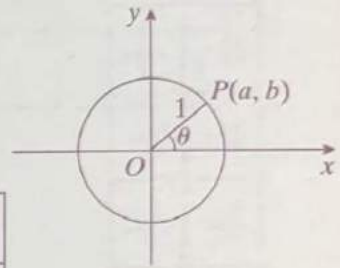
M 77 a Graphs of Trigonometric Functions 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1~

Given a unit circle (a circle with radius 1) with center at the origin, letting $P(a, b)$ be a point on the circle, and letting θ be the shortest angle formed counterclockwise from the positive x-axis to OP,

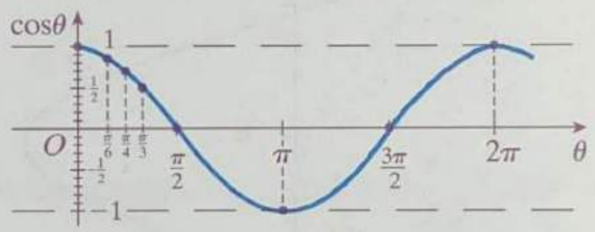
$$\cos\theta = \frac{a}{1} = \boxed{a}$$



Reviewing the values of $\cos\theta$ for some common angles:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1

Plotting the above values, trace the graph of $\cos\theta$ from 0 to 2π.

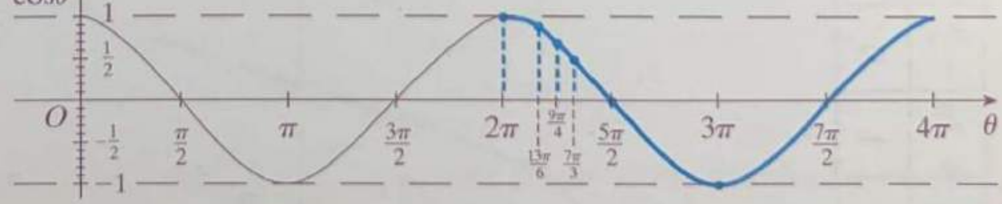


Answers: $a, \frac{\sqrt{2}}{2}, \frac{1}{2}, 0, -1, 0, 1$

Complete and check your answers.

θ	2π	$\frac{13\pi}{6}$	$\frac{9\pi}{4}$	$\frac{7\pi}{3}$	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1

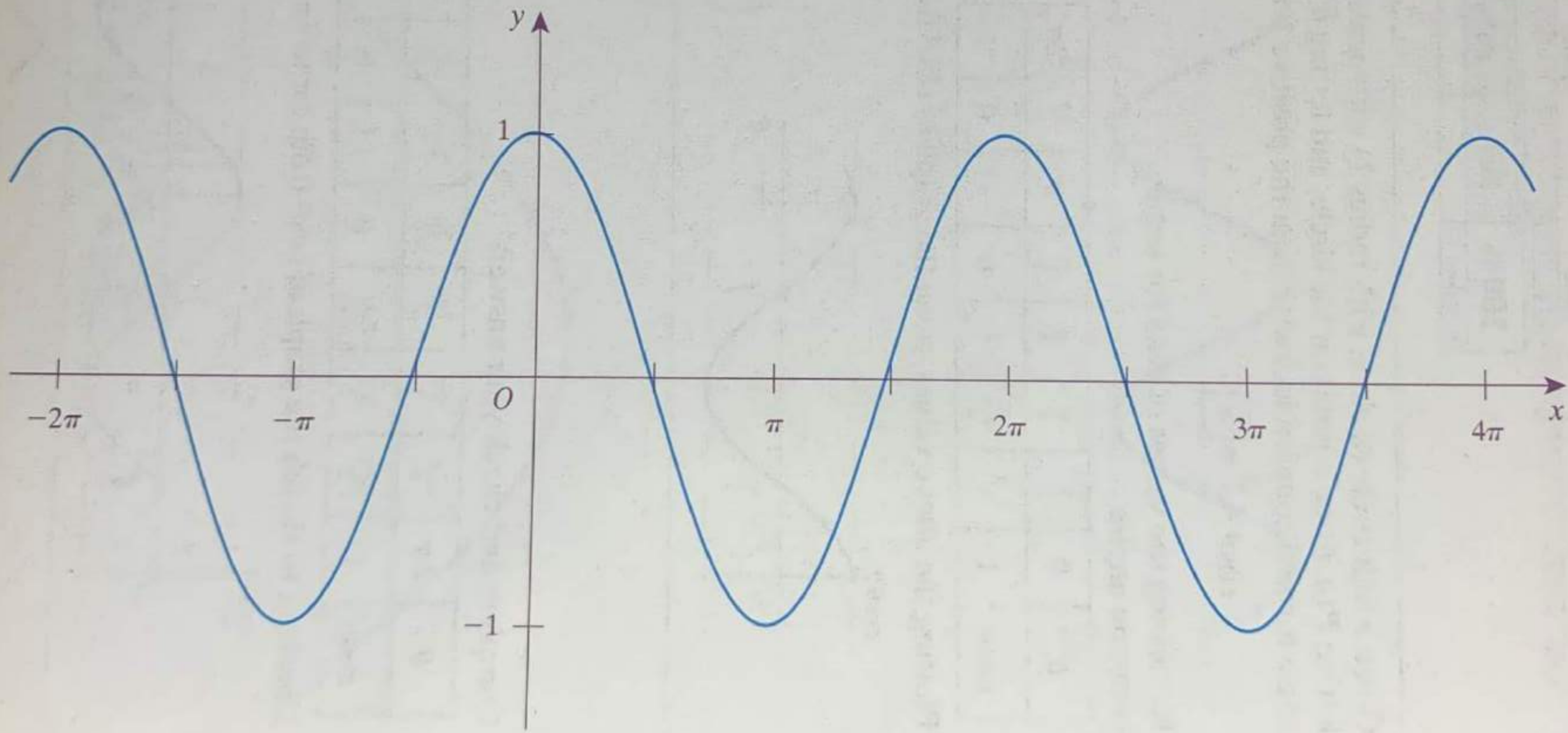
Continue to sketch the graph of $\cos\theta$ from 2π to 4π.



Answers: $1, \frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}, 0, -1, 0, 1$

1. Draw the graph of $y = \cos x$ from -2π to 4π.

x	-2π	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{13\pi}{6}$	$\frac{9\pi}{4}$	$\frac{7\pi}{3}$	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
y	1	0	-1	0	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1



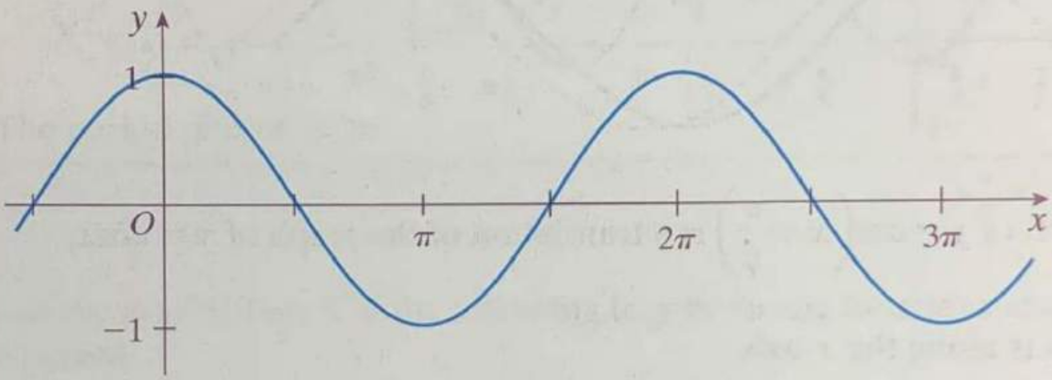
M 78 a Graphs of Trigonometric Functions 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

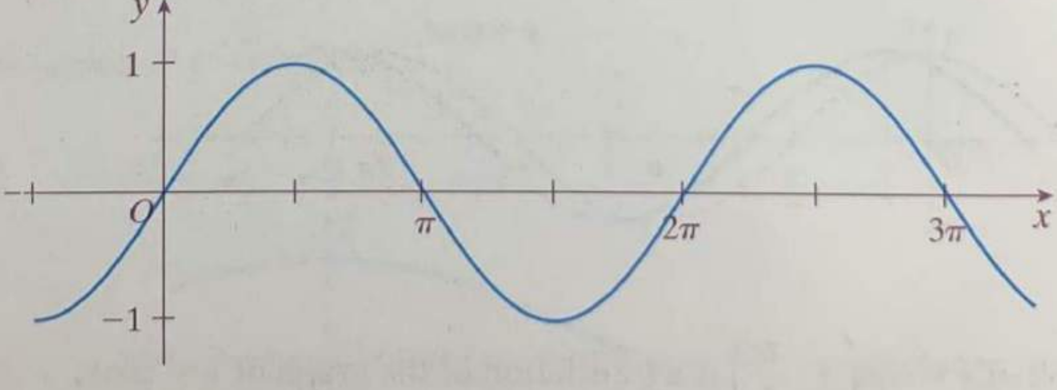
1. Draw the graph of $y = \cos x$ from $-\frac{\pi}{2}$ to 3π.

x	$-\frac{\pi}{2}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π
y	0	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1	0	-1



2. Draw the graph of $y = \cos\left(x - \frac{\pi}{2}\right)$ from $-\frac{\pi}{2}$ to 3π.

x	$-\frac{\pi}{2}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π
y	-1	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0	1	0



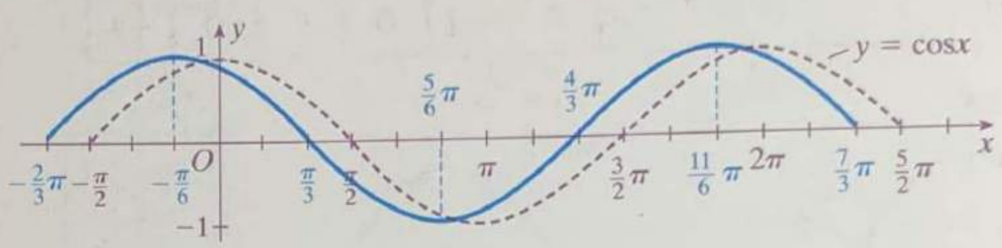
The graph of $y = \cos\left(x - \frac{\pi}{2}\right)$ is a translation of the graph of $y = \cos x$, $\boxed{\frac{\pi}{2}}$ unit(s) along the $\boxed{x}$-axis.

M 78 b

3. Use the graph of $y = \cos x$ as a reference to draw the graph of each of the following trigonometric functions.

(1) $y=\cos\left(x+\frac{\pi}{6}\right)$

x	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
y	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$

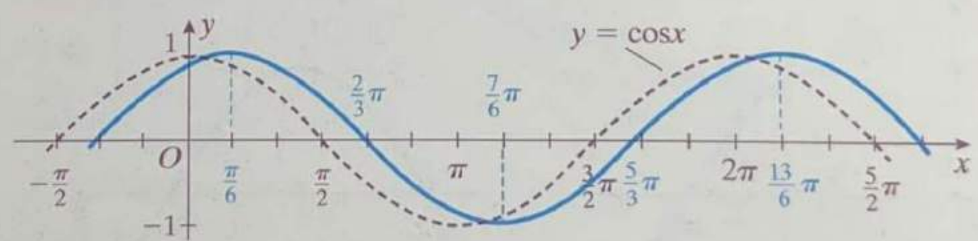


The graph of $y=\cos\left(x+\frac{\pi}{6}\right)$ is a translation of the graph of $y = \cos x$,

$\boxed{-\frac{\pi}{6}}$ units along the x-axis.

(2) $y=\cos\left(x-\frac{\pi}{6}\right)$

x	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
y	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$



The graph of $y=\cos\left(x-\frac{\pi}{6}\right)$ is a translation of the graph of $y = \cos x$,

$\boxed{\frac{\pi}{6}}$ units along the x-axis.

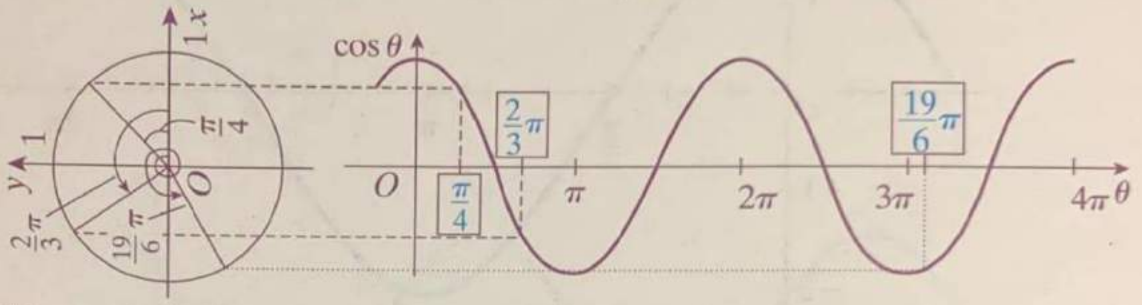
KUMON

M 79 a Graphs of Trigonometric Functions 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

Notice below how the cosine curve stems from the graph of the cosine function between 0 and 2π on the unit circle.

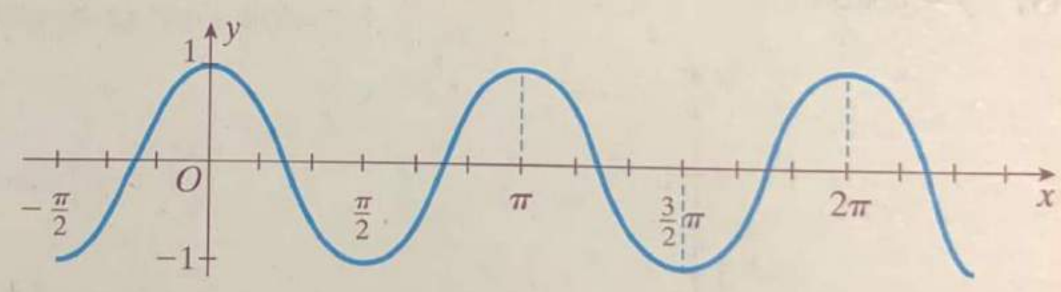


The period of $\cos\theta$ is 2π.

Answers: $\frac{\pi}{4}, \frac{2}{3}\pi, \frac{19}{6}\pi$

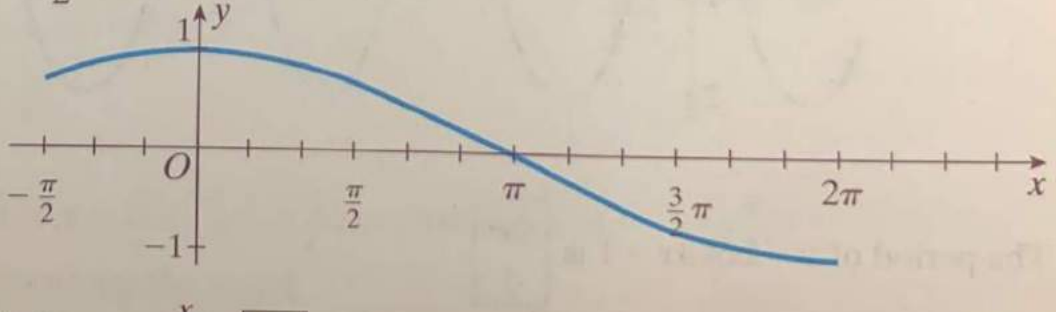
Draw the graph of each of the following trigonometric functions and state the period.

(1) $y = \cos 2x$



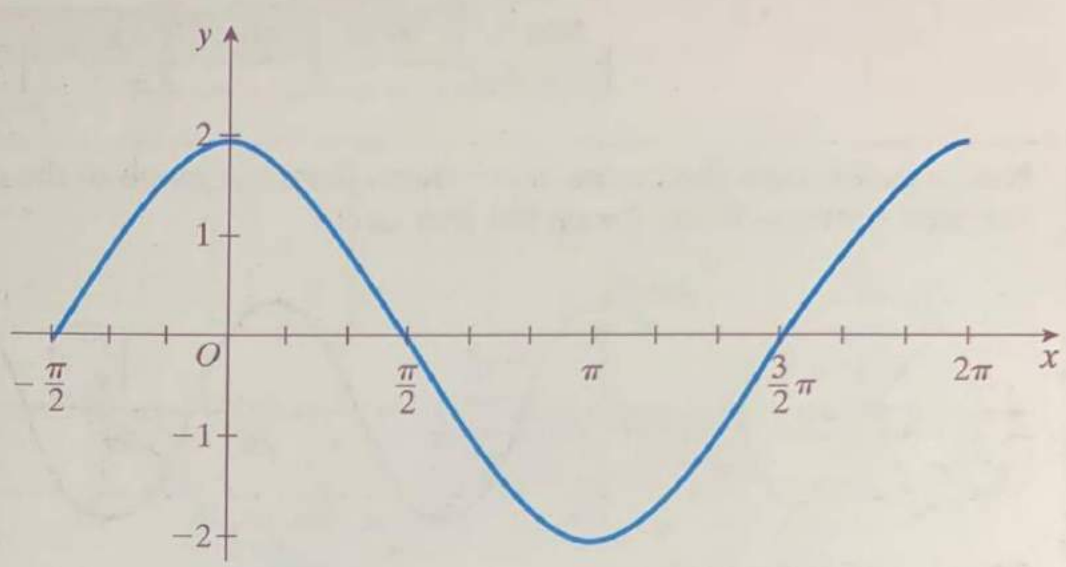
The period of $y = \cos 2x$ is $\boxed{\pi}$.

(2) $y = \cos\frac{x}{2}$



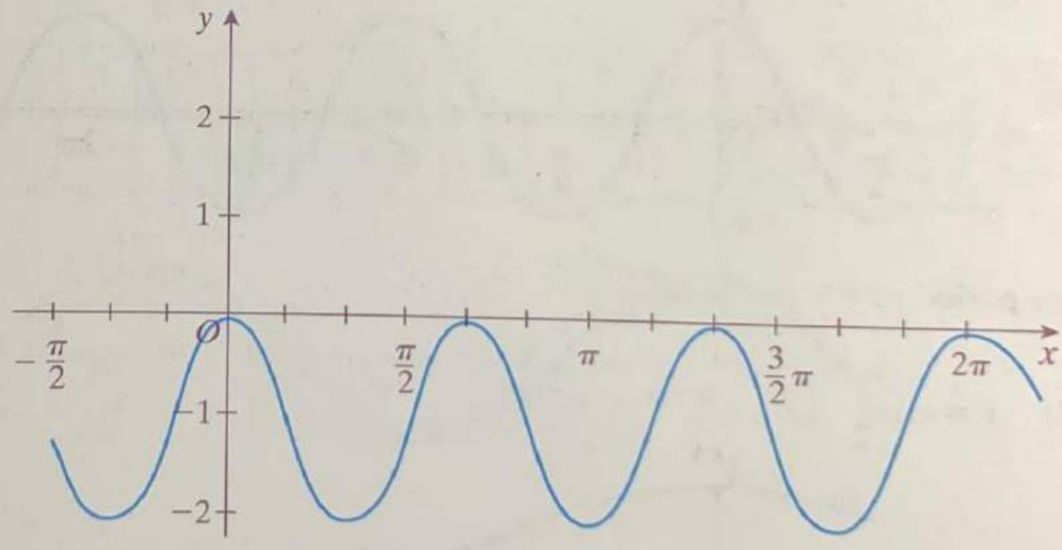
The period of $y = \cos\frac{x}{2}$ is $\boxed{4\pi}$.

(3) $y = 2\cos x$



The period of $y = 2\cos x$ is $\boxed{2\pi}$.

(4) $y = \cos 3x - 1$



The period of $y = \cos 3x - 1$ is $\boxed{\frac{2\pi}{3}}$.

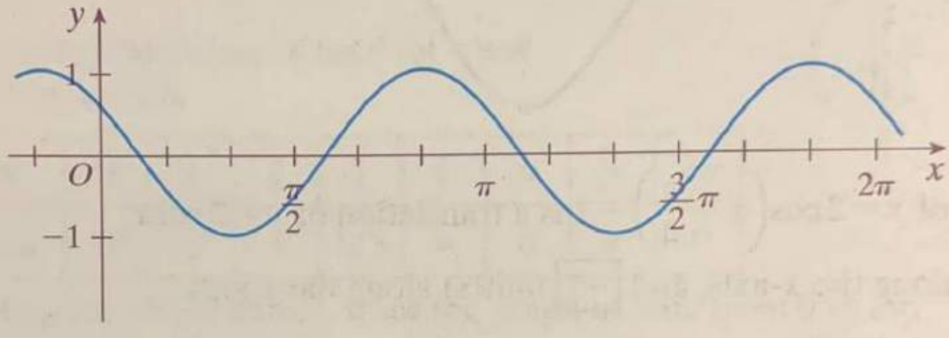
Graphs of Trigonometric Functions 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

Draw the graph of each of the following trigonometric functions and state the translation.

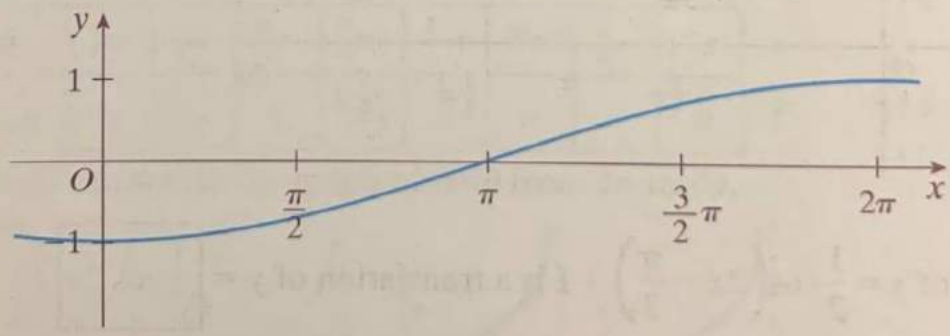
(1) $y = \cos\left(2x + \frac{\pi}{3}\right)$



The graph of $y = \cos\left(2x + \frac{\pi}{3}\right)$ is a translation of $y = \cos 2x$,

$\boxed{-\frac{\pi}{6}}$ units along the x-axis.

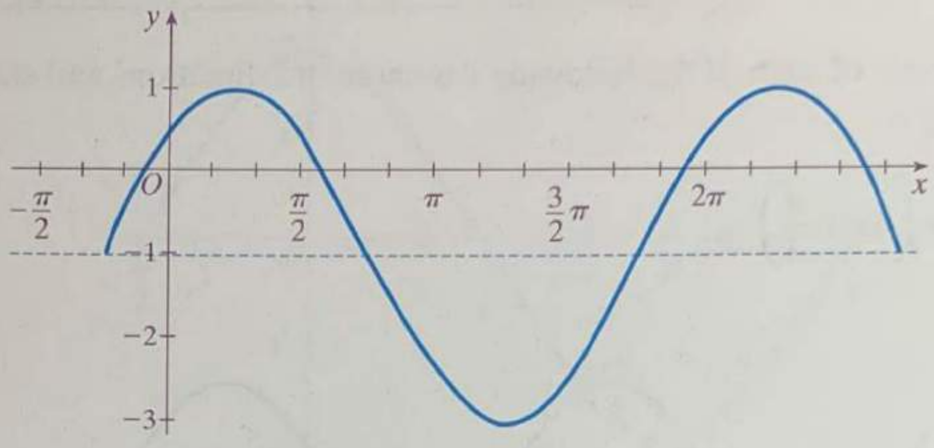
(2) $y = \cos\left(\frac{x}{2} + \pi\right)$



The graph of $y = \cos\left(\frac{x}{2} + \pi\right)$ is a translation of $y = \cos\frac{x}{2}$,

$\boxed{-2\pi}$ units along the x-axis.

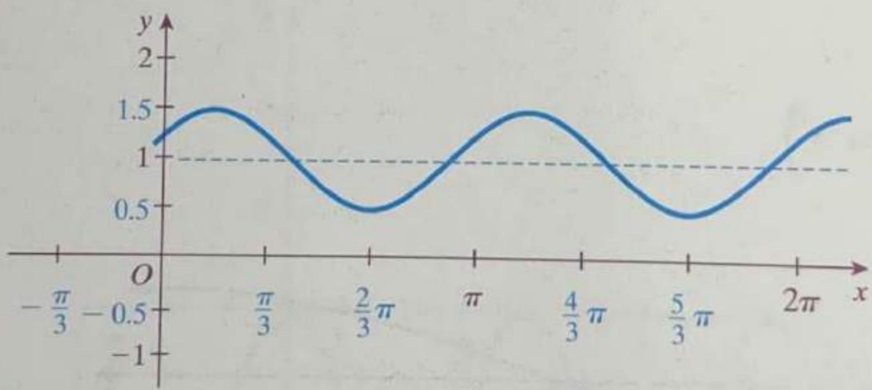
(3) $y = 2\cos\left(x - \frac{\pi}{4}\right) - 1$



The graph of $y = 2\cos\left(x - \frac{\pi}{4}\right) - 1$ is a translation of $y = 2\cos x$,

$\boxed{\frac{\pi}{4}}$ units along the x-axis, and $\boxed{-1}$ unit(s) along the y-axis.

(4) $y = \frac{1}{2}\cos\left(2x - \frac{\pi}{3}\right) + 1$



The graph of $y = \frac{1}{2}\cos\left(2x - \frac{\pi}{3}\right) + 1$ is a translation of $y = \boxed{\frac{1}{2}\cos 2x}$,

$\boxed{\frac{\pi}{6}}$ units along the x-axis, and $\boxed{1}$ unit(s) along the y-axis.

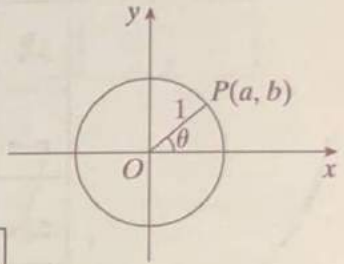
M 81 a Graphs of Trigonometric Functions 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1~

Given a unit circle (a circle with radius 1) with center at the origin, letting $P(a, b)$ be a point on the circle, and letting θ be the shortest angle formed counterclockwise from the positive x-axis to OP,

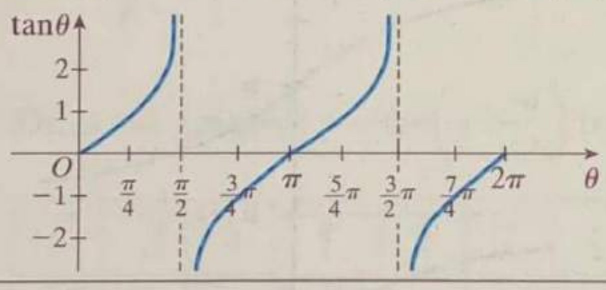
$$\tan\theta = \frac{b}{a}$$



Reviewing the values of $\tan\theta$ for some common angles:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	0	∞	0

Plotting the above values, trace the graph of $\tan\theta$ from 0 to 2π.

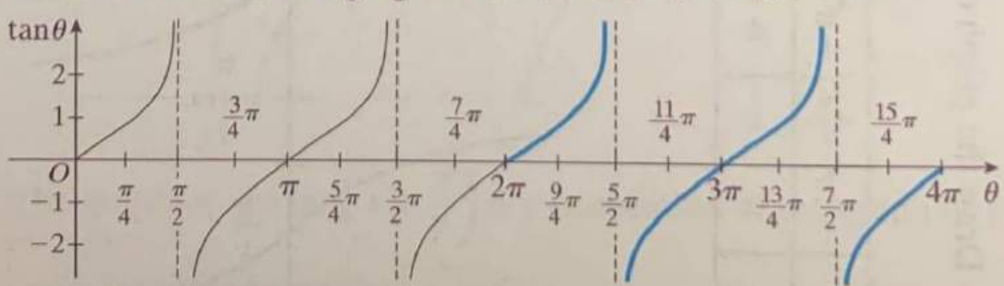


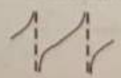
Answers: $\frac{\sqrt{3}}{3}$, 1, $\sqrt{3}$, 0, ∞, 0

Complete and check your answers.

θ	2π	$\frac{13}{6}\pi$	$\frac{9}{4}\pi$	$\frac{7}{3}\pi$	$\frac{5}{2}\pi$	3π	$\frac{7}{2}\pi$	4π
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	0	∞	0

Continue to sketch the graph of $\tan\theta$ from 2π to 4π.



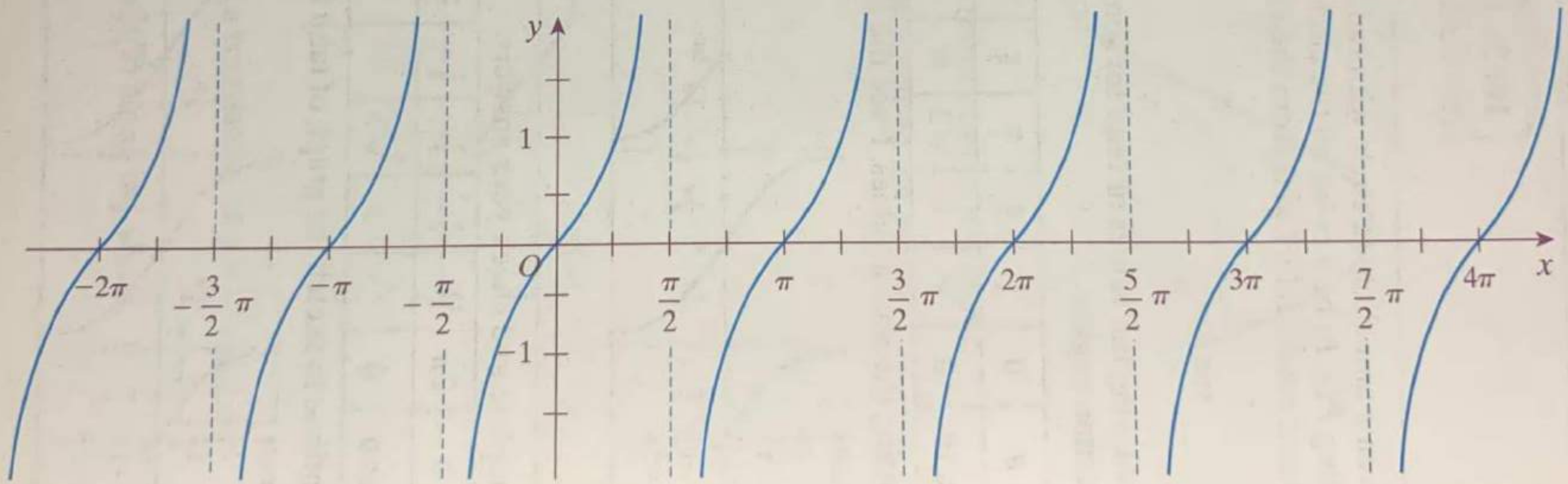


Answers: 0, $\frac{\sqrt{3}}{3}$, 1, $\sqrt{3}$, ∞, 0, ∞, 0

M 81 b

1. Draw the graph of $y = \tan x$ from -2π to 4π.

x	-2π	$-\frac{3}{2}\pi$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π	$\frac{13}{6}\pi$	$\frac{9}{4}\pi$	$\frac{7}{3}\pi$	$\frac{5}{2}\pi$	3π	$\frac{7}{2}\pi$	4π
y	0	∞	0	∞	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	0	∞	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	0	∞	0



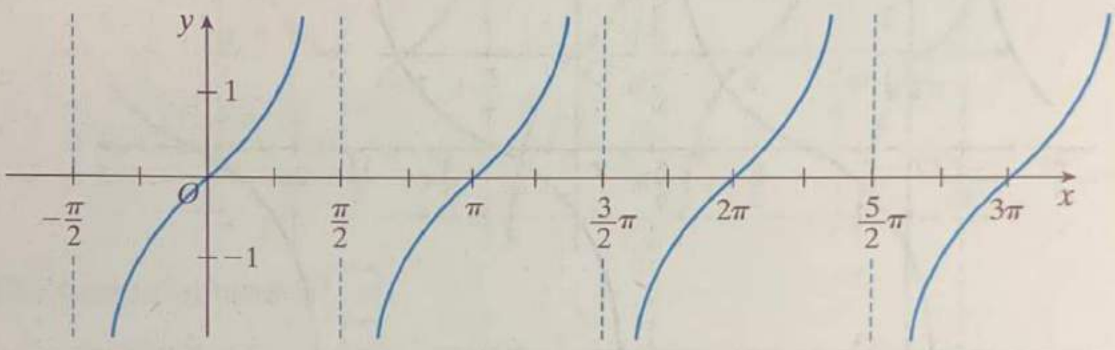
M 82 a Graphs of Trigonometric Functions 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

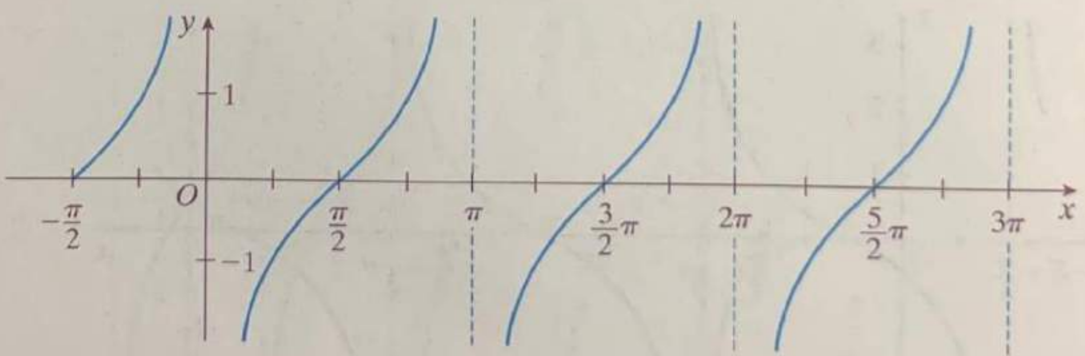
1. Draw the graph of $y = \tan x$ from $-\frac{\pi}{2}$ to 3π.

x	$-\frac{\pi}{2}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π	$\frac{5}{2}\pi$	3π
y	∞	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	0	∞	0	∞	0



2. Draw the graph of $y = \tan\left(x + \frac{\pi}{2}\right)$ from $-\frac{\pi}{2}$ to 3π.

x	$-\frac{\pi}{2}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π	$\frac{5}{2}\pi$	3π
y	0	∞	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	∞	0	∞	0	∞



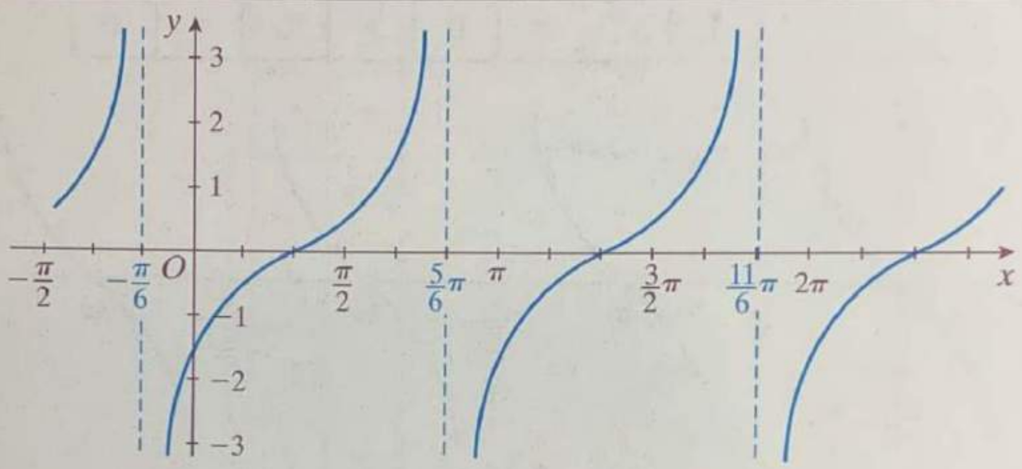
The graph of $y = \tan\left(x + \frac{\pi}{2}\right)$ is a translation of the graph of $y = \tan x$, $\boxed{-\frac{\pi}{2}}$ unit(s) along the $\boxed{x}$-axis.

M 82 b

3. Draw the graph of each of the following trigonometric functions, and state the translation.

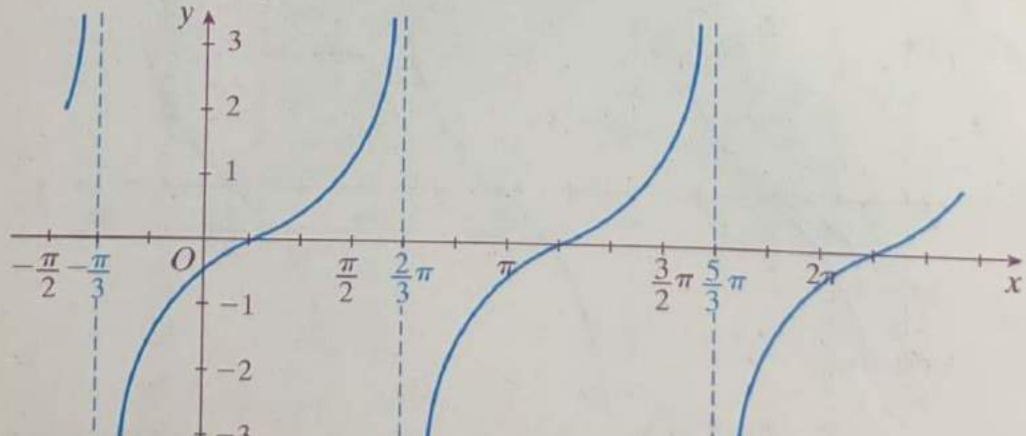
(1) $y=\tan\left(x-\frac{\pi}{3}\right)$

x	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2}{3}\pi$	$\frac{5}{6}\pi$	π	$\frac{7}{6}\pi$	$\frac{4}{3}\pi$	$\frac{3}{2}\pi$
y	∞	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	∞	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$



The graph of $y=\tan\left(x-\frac{\pi}{3}\right)$ is a translation of the graph of $y=\tan x$, $\boxed{\frac{\pi}{3}}$ unit(s) along the $\boxed{x}$-axis.

(2) $y=\tan\left(x-\frac{\pi}{6}\right)$



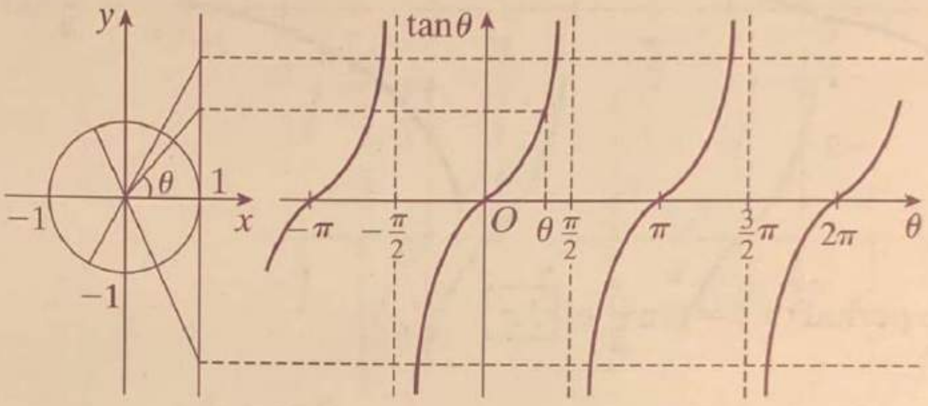
The graph of $y=\tan\left(x-\frac{\pi}{6}\right)$ is a translation of the graph of $y=\tan x$, $\boxed{\frac{\pi}{6}}$ unit(s) along the $\boxed{x}$-axis.

Graphs of Trigonometric Functions 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1~

Notice below how the tangent curve stems from the graph of the tangent function between 0 and 2π on the unit circle.

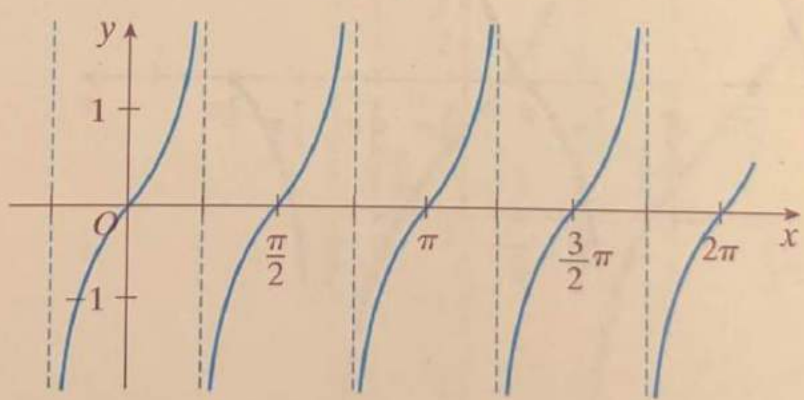


The period of $\tan\theta$ is $\boxed{\pi}$.

Answer: π

Draw the graph of each of the following trigonometric functions, and state the period.

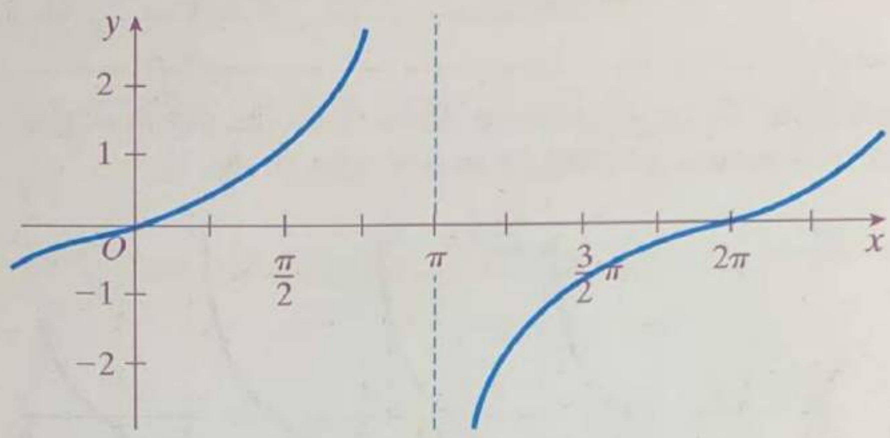
(1) $y = \tan 2x$



The period of $y = \tan 2x$ is $\boxed{\frac{\pi}{2}}$.

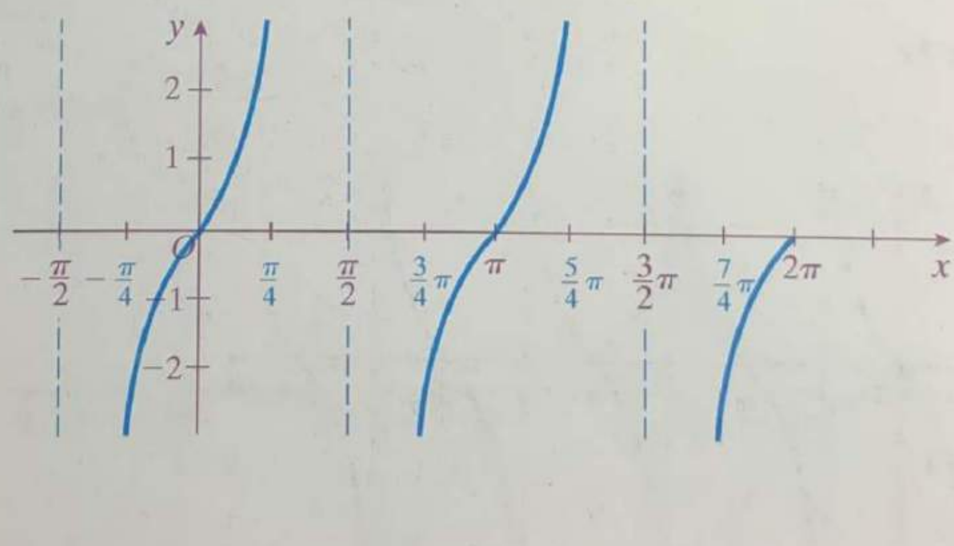
M 83 b

(2) $y = \tan \frac{x}{2}$



The period of $y = \tan \frac{x}{2}$ is $\boxed{2\pi}$.

(3) $y = 2\tan x$



The period of $y = 2\tan x$ is $\boxed{\pi}$.

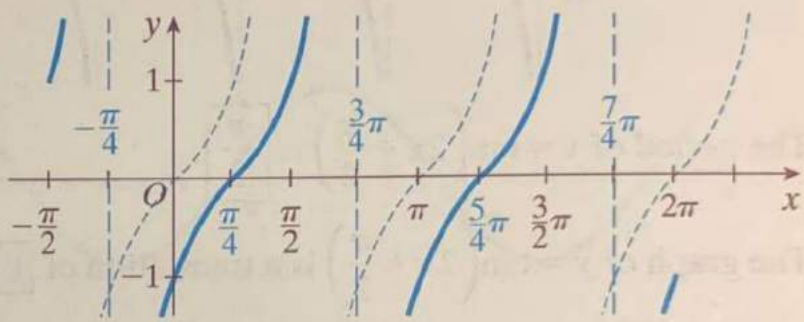
M 84 a Graphs of Trigonometric Functions 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

Draw the graph of each of the following trigonometric functions, state the period, and fill in the blank boxes to state the translation.

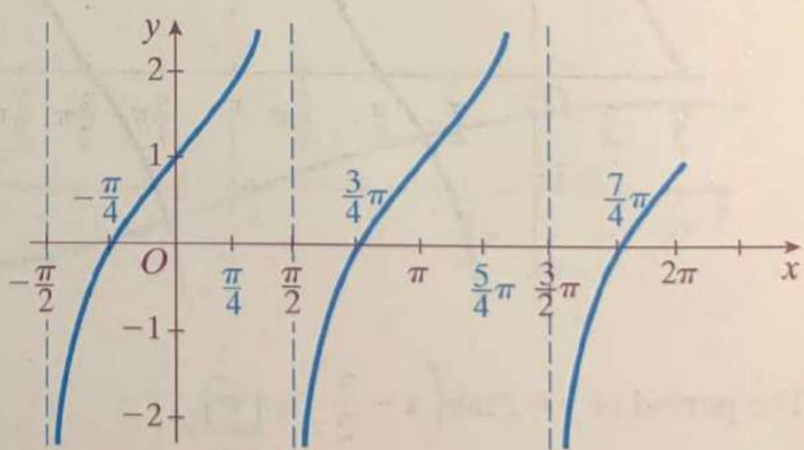
(1) $y=\tan\left(x-\frac{\pi}{4}\right)$



The period of $y=\tan\left(x-\frac{\pi}{4}\right)$ is $\boxed{\pi}$.

The graph of $y=\tan\left(x-\frac{\pi}{4}\right)$ is a translation of $\boxed{y=\tan x}$, $\boxed{\frac{\pi}{4}}$ units along the x-axis.

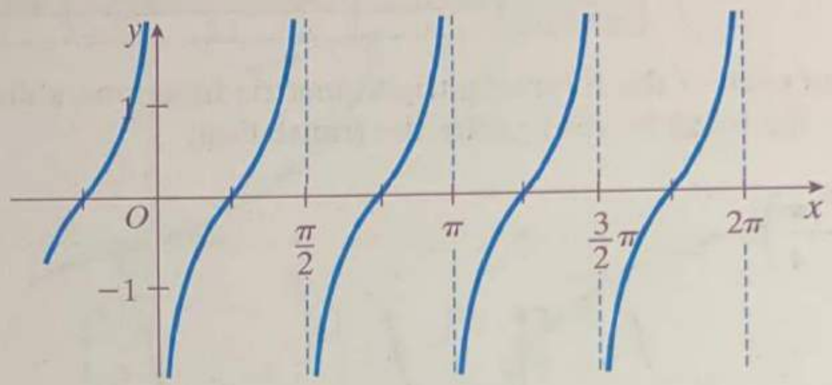
(2) $y=\tan x+1$



The period of $y=\tan x+1$ is $\boxed{\pi}$.
The graph of $y=\tan x+1$ is a translation of $\boxed{y=\tan x}$, $\boxed{1}$ unit(s) along the $\boxed{y}$ -axis.

M 84 b

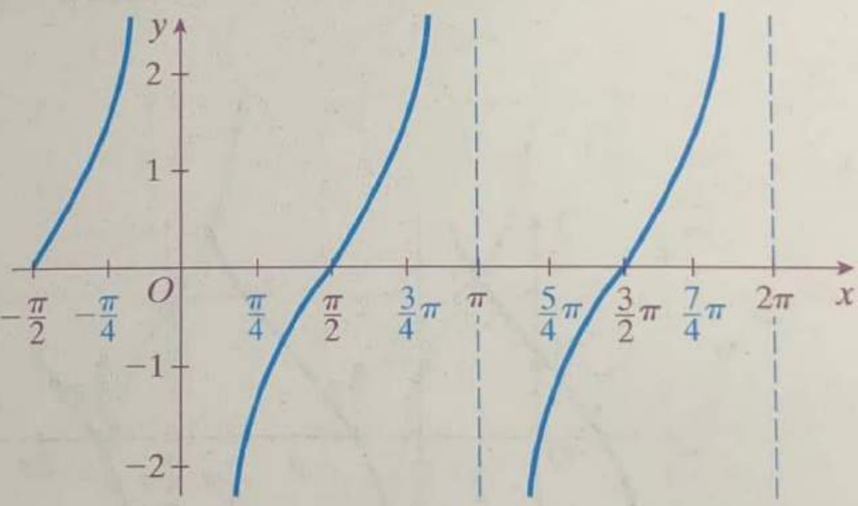
(3) $y = \tan\left(2x + \frac{\pi}{2}\right)$



The period of $y = \tan\left(2x + \frac{\pi}{2}\right)$ is $\boxed{\frac{\pi}{2}}$.

The graph of $y = \tan\left(2x + \frac{\pi}{2}\right)$ is a translation of $\boxed{y = \tan 2x}$, $\boxed{-\frac{\pi}{4}}$ units along the $\boxed{x}$ -axis.

(4) $y = 2\tan\left(x - \frac{\pi}{2}\right)$



The period of $y = 2\tan\left(x - \frac{\pi}{2}\right)$ is $\boxed{\pi}$.

The graph of $y = 2\tan\left(x - \frac{\pi}{2}\right)$ is a translation of $\boxed{y = 2\tan x}$, $\boxed{\frac{\pi}{2}}$ units along the $\boxed{x}$ -axis.

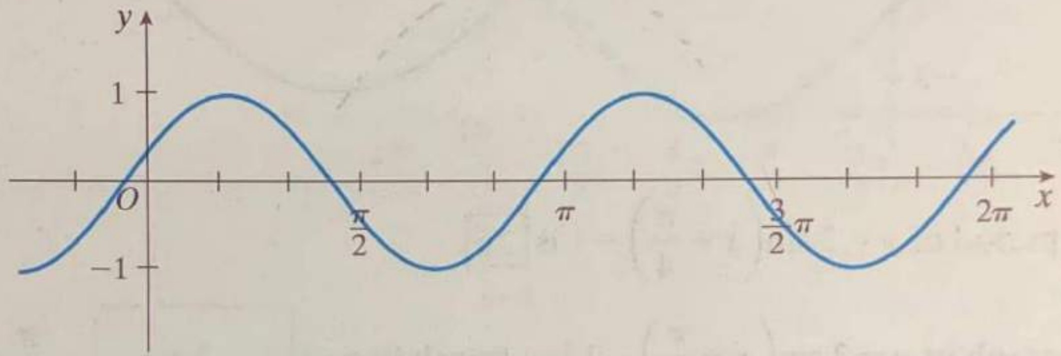
M 85 a Graphs of Trigonometric Functions 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

In each of the following exercises, draw the graph of the given function and state the period and translation.

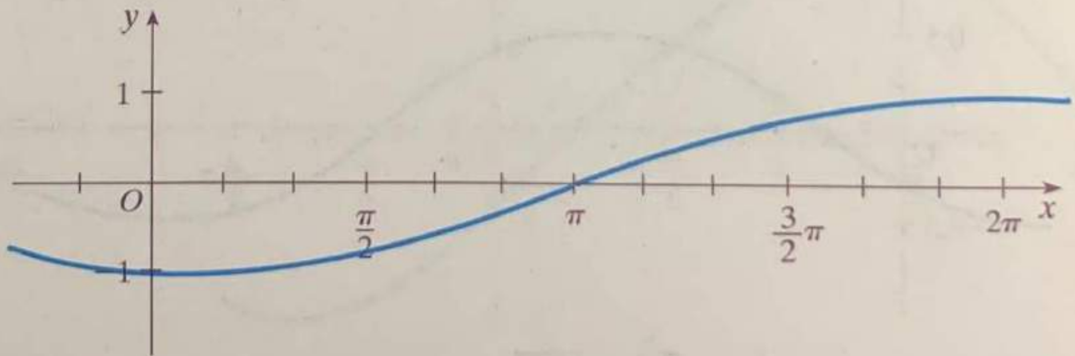
(1) $y=\sin\left(2x+\frac{\pi}{6}\right)$



The period of $y=\sin\left(2x+\frac{\pi}{6}\right)$ is $\boxed{\pi}$.

The graph of $y=\sin\left(2x+\frac{\pi}{6}\right)$ is a translation of $\boxed{y=\sin 2x}$, $-\frac{\pi}{12}$ units along the x-axis.

(2) $y=\sin\left(\frac{x}{2}-\frac{\pi}{2}\right)$

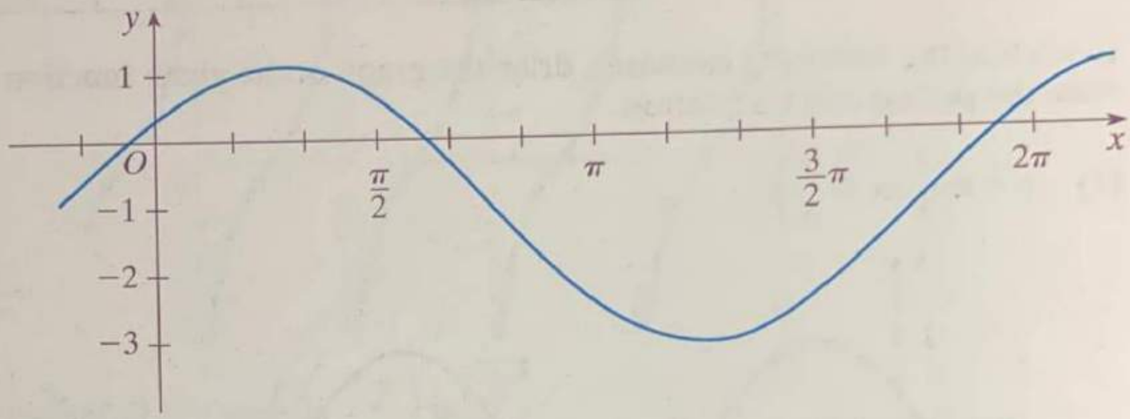


The period of $y=\sin\left(\frac{x}{2}-\frac{\pi}{2}\right)$ is $\boxed{4\pi}$.

The graph of $y=\sin\left(\frac{x}{2}-\frac{\pi}{2}\right)$ is a translation of $\boxed{y=\sin\frac{x}{2}}$, π units along the x-axis.

M 85 b

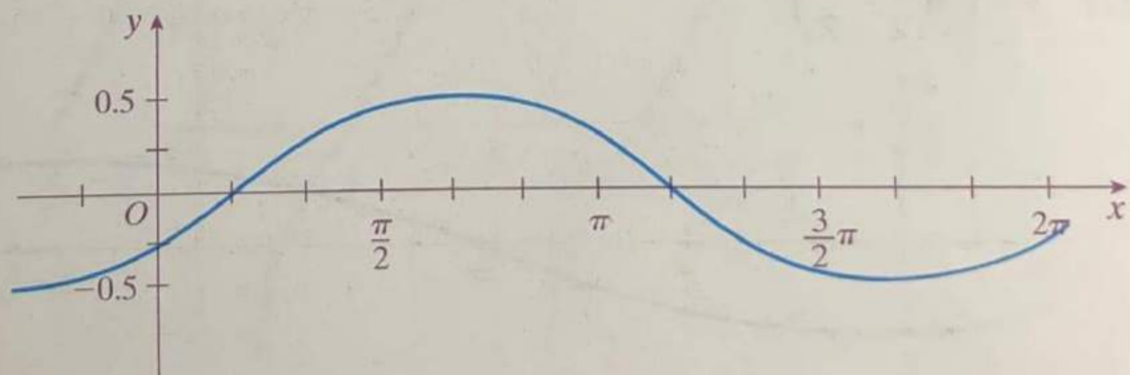
(3) $y=2\sin\left(x+\frac{\pi}{4}\right)-1$



The period of $y=2\sin\left(x+\frac{\pi}{4}\right)-1$ is $\boxed{2\pi}$.

The graph of $y=2\sin\left(x+\frac{\pi}{4}\right)-1$ is a translation of $\boxed{y=2\sin x}$, $-\frac{\pi}{4}$ units along the x-axis and -1 unit along the y-axis.

(4) $y=\frac{1}{2}\sin\left(x-\frac{\pi}{6}\right)$



The period of $y=\frac{1}{2}\sin\left(x-\frac{\pi}{6}\right)$ is $\boxed{2\pi}$.

The graph of $y=\frac{1}{2}\sin\left(x-\frac{\pi}{6}\right)$ is a translation of $\boxed{y=\frac{1}{2}\sin x}$, $\frac{\pi}{6}$ units along the x-axis

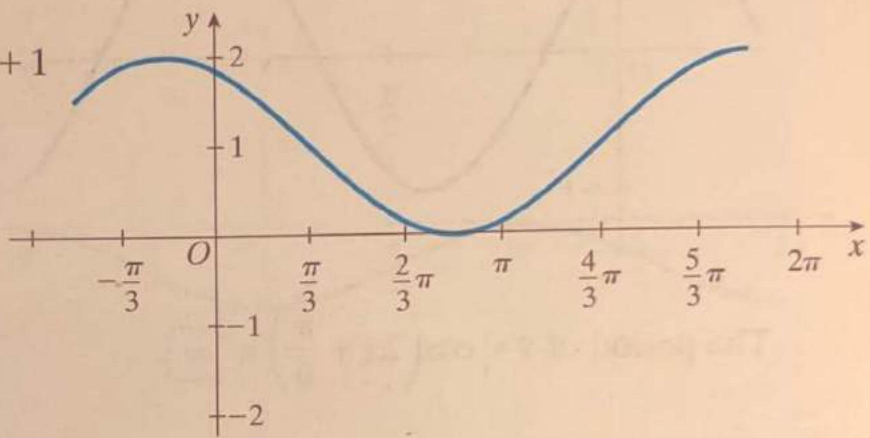
M 86 a Graphs of Trigonometric Functions 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

In each of the following exercises, draw the graph of the given function and state the period and translation.

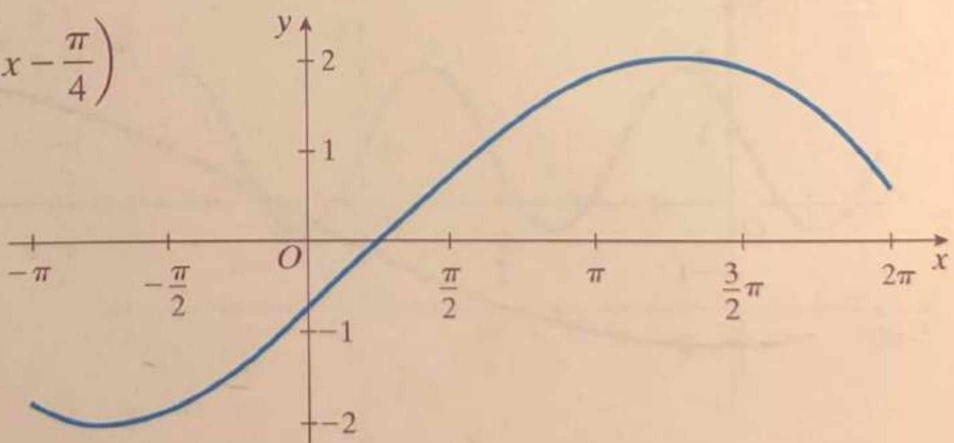
(1) $y=\sin\left(x+\frac{2}{3}\pi\right)+1$



The period of $y=\sin\left(x+\frac{2}{3}\pi\right)+1$ is $\boxed{2\pi}$.

The graph of $y=\sin\left(x+\frac{2}{3}\pi\right)+1$ is a translation of $\boxed{y=\sin x}$, $-\frac{2}{3}\pi$ units along the x-axis, and 1 unit along the y-axis.

(2) $y=2\sin\frac{1}{2}\left(x-\frac{\pi}{4}\right)$

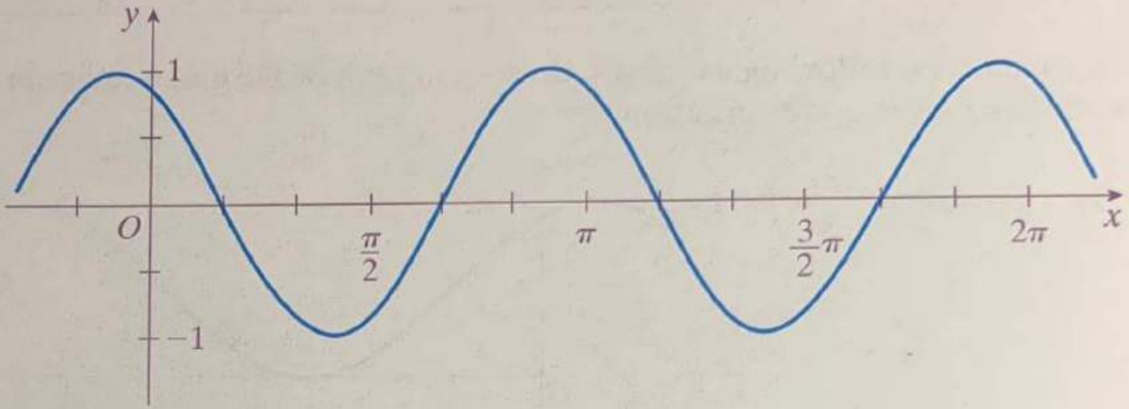


The period of $y=2\sin\frac{1}{2}\left(x-\frac{\pi}{4}\right)$ is $\boxed{4\pi}$.

The graph of $y=2\sin\frac{1}{2}\left(x-\frac{\pi}{4}\right)$ is a translation of $\boxed{y=2\sin\frac{x}{2}}$, $\frac{\pi}{4}$ units along the x-axis.

M 86 b

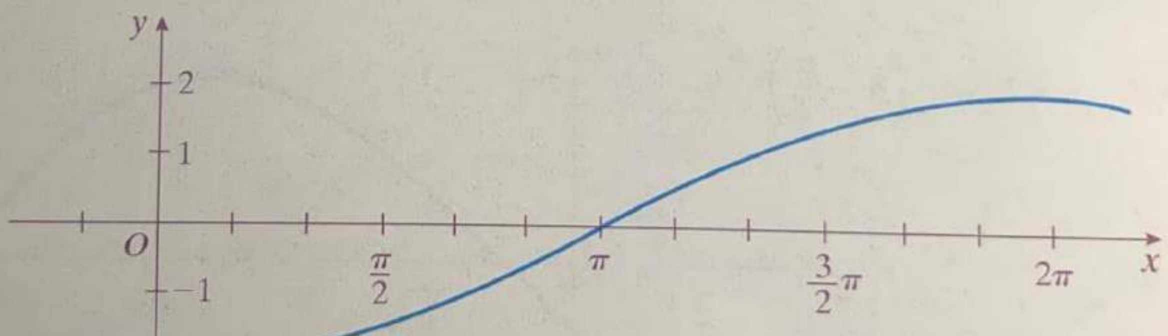
(3) $y = \cos\left(2x + \dfrac{\pi}{6}\right)$



The period of $y = \cos\left(2x + \dfrac{\pi}{6}\right)$ is $\boxed{\pi}$.

The graph of $y = \cos\left(2x + \dfrac{\pi}{6}\right)$ is a translation of $\boxed{y = \cos 2x}$, $-\dfrac{\pi}{12}$ units along the x-axis.

(4) $y = 2\cos\left(\dfrac{x}{2} - \pi\right)$



The period of $y = 2\cos\left(\dfrac{x}{2} - \pi\right)$ is $\boxed{4\pi}$.

The graph of $y = 2\cos\left(\dfrac{x}{2} - \pi\right)$ is a translation of $\boxed{y = 2\cos\dfrac{x}{2}}$, 2π units along the x-axis.

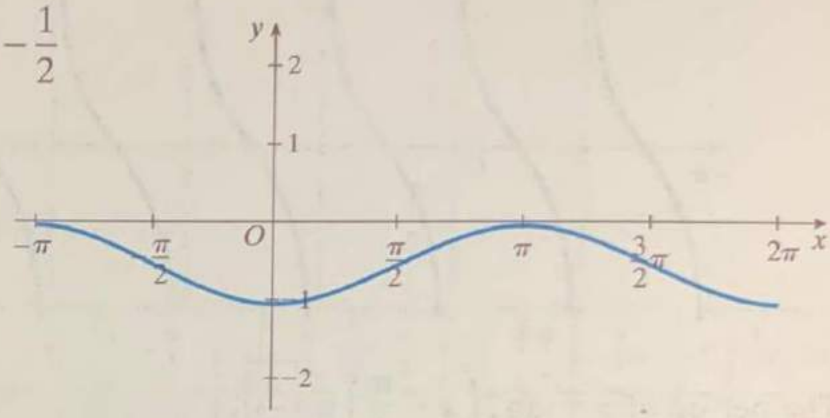
Graphs of Trigonometric Functions 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

1. In each of the following exercises, draw the graph of the given function and state the period and translation.

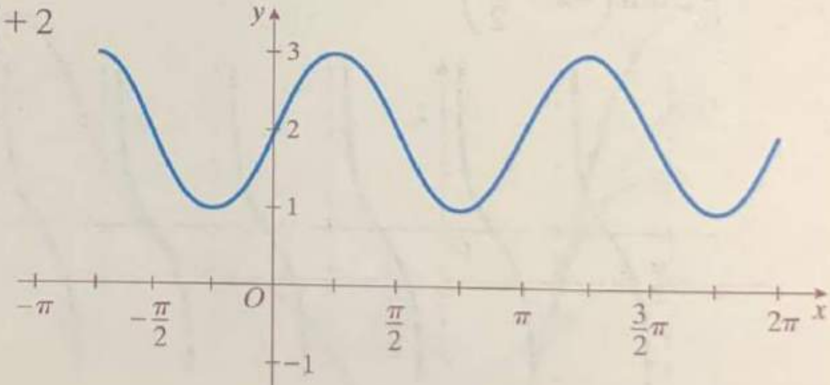
(1) $y=\frac{1}{2}\cos(x-\pi)-\frac{1}{2}$



The period of $y=\frac{1}{2}\cos(x-\pi)-\frac{1}{2}$ is $\boxed{2\pi}$.

The graph of $y=\frac{1}{2}\cos(x-\pi)-\frac{1}{2}$ is a translation of $\boxed{y=\frac{1}{2}\cos x}$, π units along the x-axis, and $-\frac{1}{2}$ units along the y-axis.

(2) $y=\cos\left(2x-\frac{\pi}{2}\right)+2$

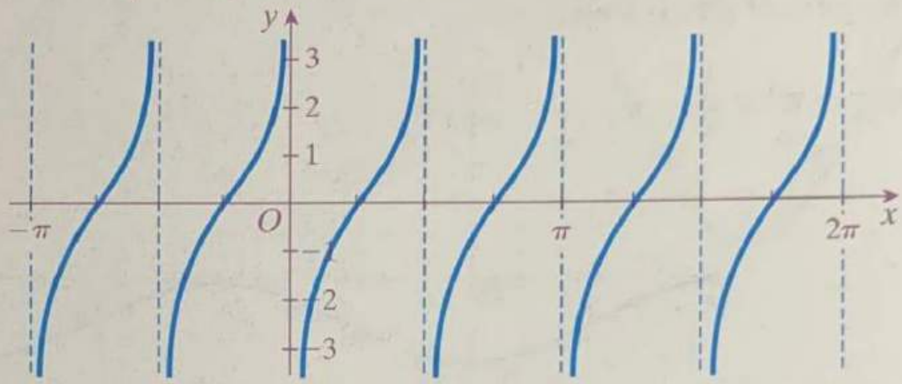


The period of $y=\cos\left(2x-\frac{\pi}{2}\right)+2$ is $\boxed{\pi}$.

The graph of $y=\cos\left(2x-\frac{\pi}{2}\right)+2$ is a translation of $\boxed{y=\cos 2x}$, $\frac{\pi}{4}$ units along the x-axis and 2 units along the y-axis.

2. In each of the following exercises, draw the graph of the given function and state the period and translation.

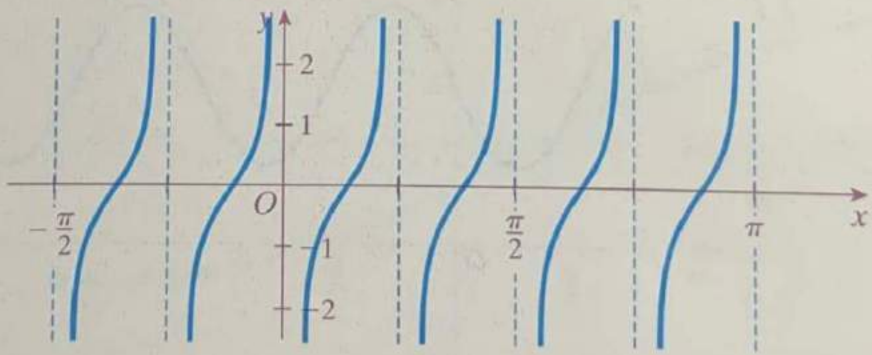
(1) $y = \tan 2\left(x - \frac{\pi}{4}\right)$



The period of $y = \tan 2\left(x - \frac{\pi}{4}\right)$ is $\boxed{\frac{\pi}{2}}$.

The graph of $y = \tan 2\left(x - \frac{\pi}{4}\right)$ is a translation of $\boxed{y = \tan 2x}$, $\frac{\pi}{4}$ units along the x-axis.

(2) $y = 2\tan\left(4x - \frac{\pi}{2}\right)$



The period of $y = 2\tan\left(4x - \frac{\pi}{2}\right)$ is $\boxed{\frac{\pi}{4}}$.

The graph of $y = 2\tan\left(4x - \frac{\pi}{2}\right)$ is a translation of $\boxed{y = 2\tan 4x}$, $\frac{\pi}{8}$ units along the x-axis.

KUMON

M 88 a Graphs of Trigonometric Functions 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1~

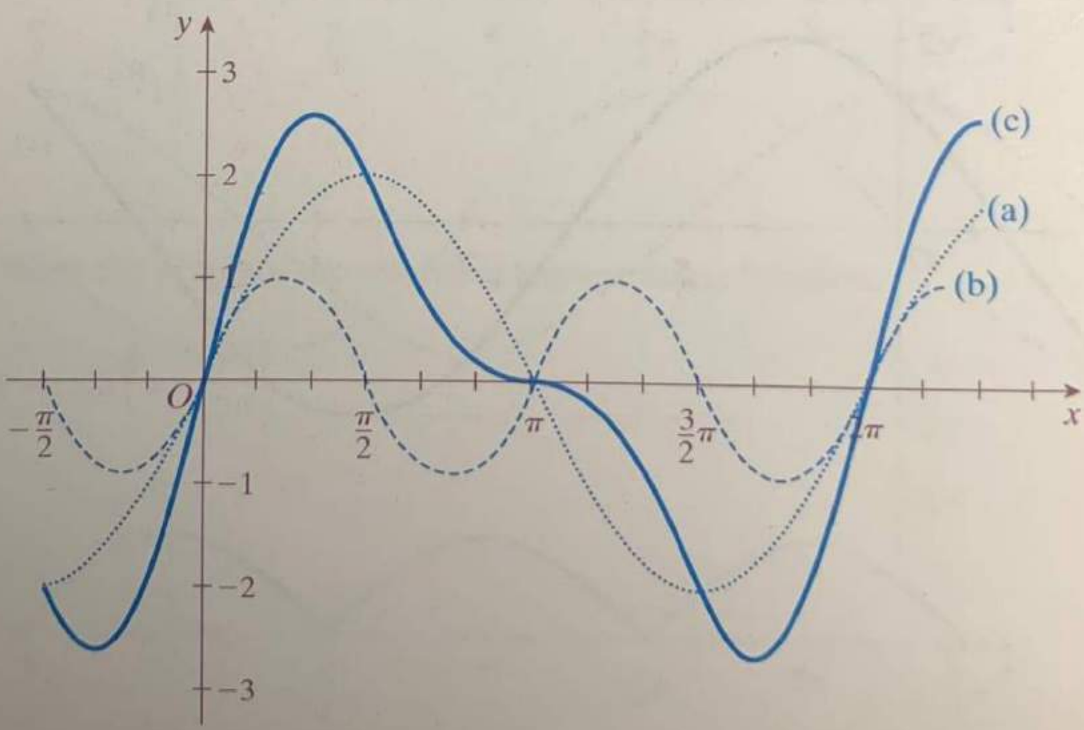
(1) Draw and label the graph of each function on the coordinate grid below.

(a) $y = 2\sin x$

(b) $y = \sin 2x$

(c) $y = 2\sin x + \sin 2x$

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{3}{2}\pi$	2π
$2\sin x$	-2	$-\sqrt{3}$	$-\sqrt{2}$	-1	0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{2}$	0	-2	0
$\sin 2x$	0	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	0	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	0	-1	0	0	0
$2\sin x+\sin 2x$	-2	$-\frac{3\sqrt{3}}{2}$	$-\sqrt{2}-1$	$\frac{-2-\sqrt{3}}{2}$	0	$\frac{2+\sqrt{3}}{2}$	$\sqrt{2}+1$	$\frac{3\sqrt{3}}{2}$	2	$\sqrt{2}-1$	0	-2	0



M 88 b

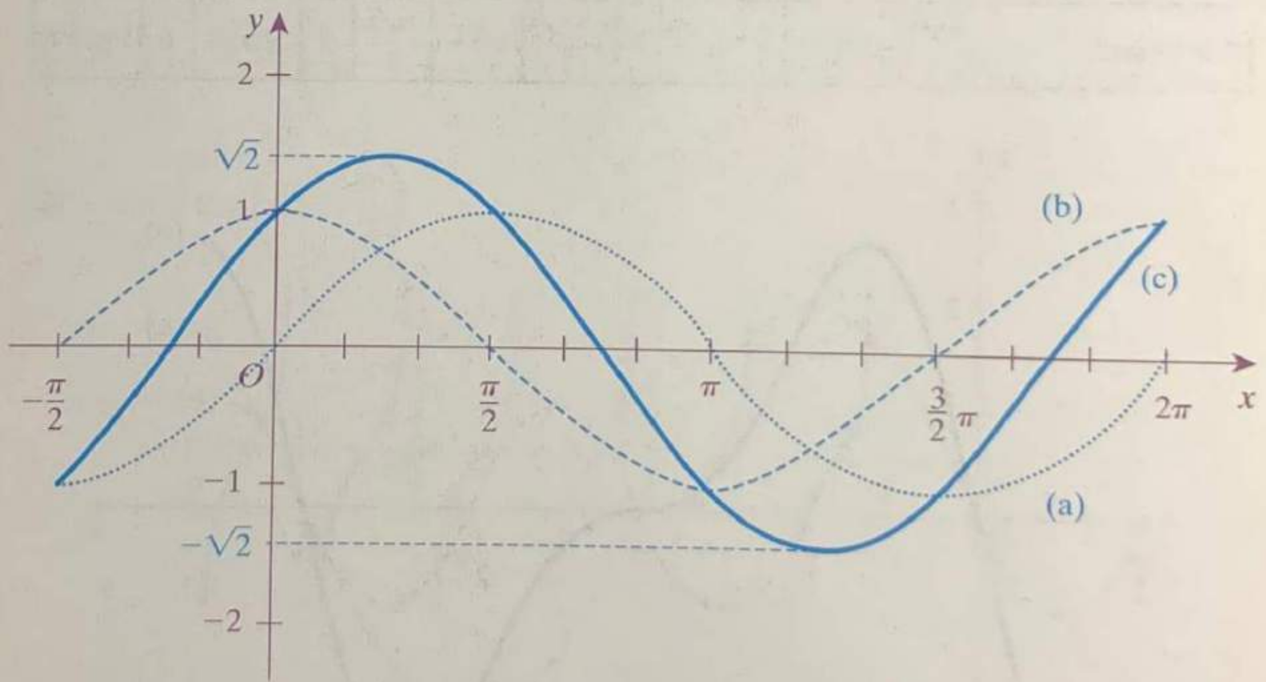
(2) Draw and label the graph of each function on the coordinate grid below.

(a) $y = \sin x$

(b) $y = \cos x$

(c) $y = \sin x + \cos x$

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	2π
$\sin x$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	$-\frac{\sqrt{2}}{2}$	-1	0
$\cos x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	$-\frac{\sqrt{2}}{2}$	0	1
$\sin x + \cos x$	-1	$\frac{-\sqrt{3}+1}{2}$	0	$\frac{-1+\sqrt{3}}{2}$	1	$\frac{1+\sqrt{3}}{2}$	$\sqrt{2}$	$\frac{\sqrt{3}+1}{2}$	1	-1	$-\sqrt{2}$	-1	1



Graphs of Trigonometric Functions 2

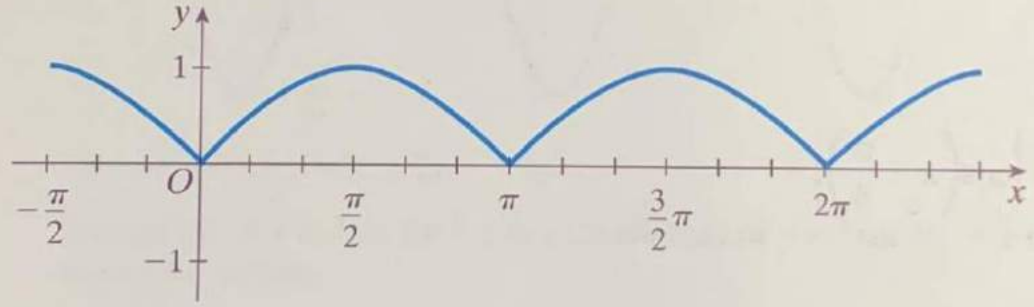
Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

1. Fill in the blanks on the chart, and draw the graph of the following trigonometric function.

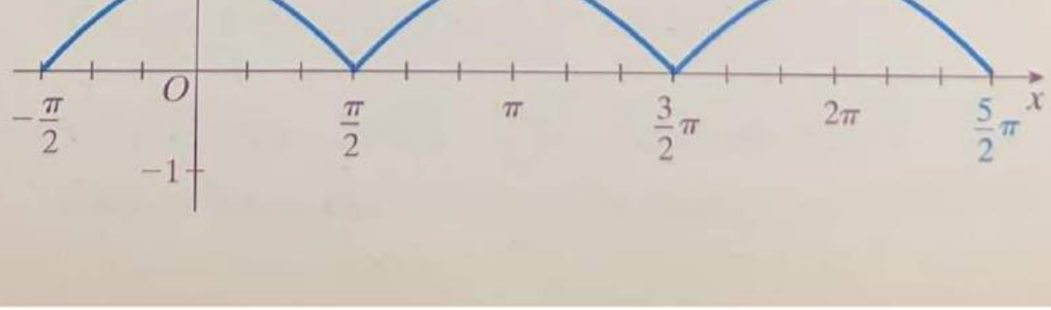
$y = |\sin x|$

x	...	π	$\frac{7}{6}\pi$	$\frac{4}{3}\pi$	$\frac{3}{2}\pi$	$\frac{5}{3}\pi$	$\frac{11}{6}\pi$	2π	...
$\sin x$	...	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	...
$\lvert\sin x\rvert$	...	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	...



2. Draw the graph of the following trigonometric function.

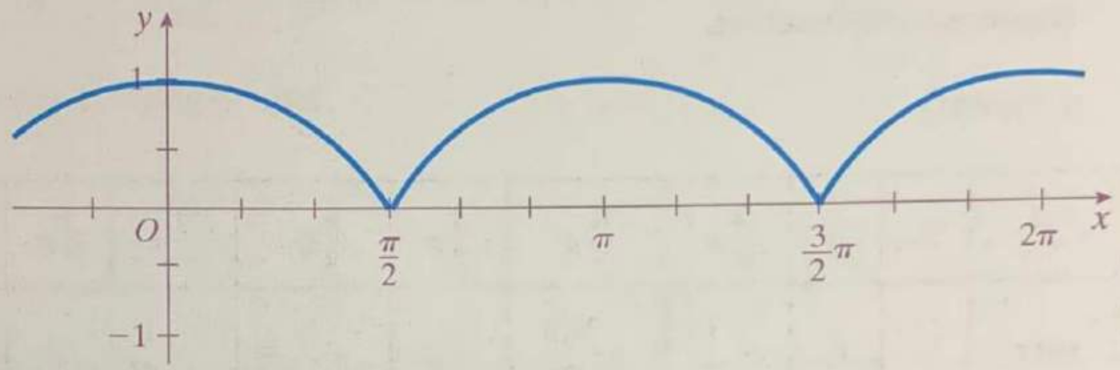
$y = \left|\sin\left(x - \frac{\pi}{2}\right)\right|$



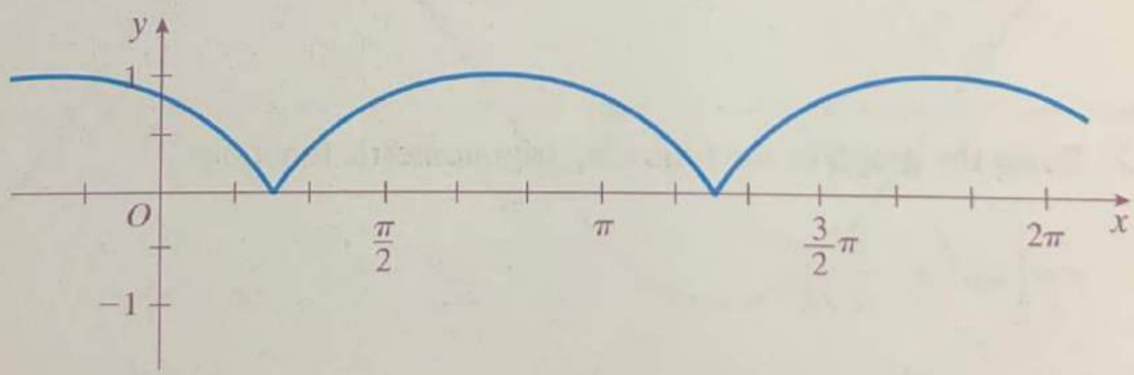
M 89 b

3. Draw the graph of each of the following trigonometric functions.

(1) $y = |\cos x|$



(2) $y = \left|\cos\left(x + \frac{\pi}{4}\right)\right|$



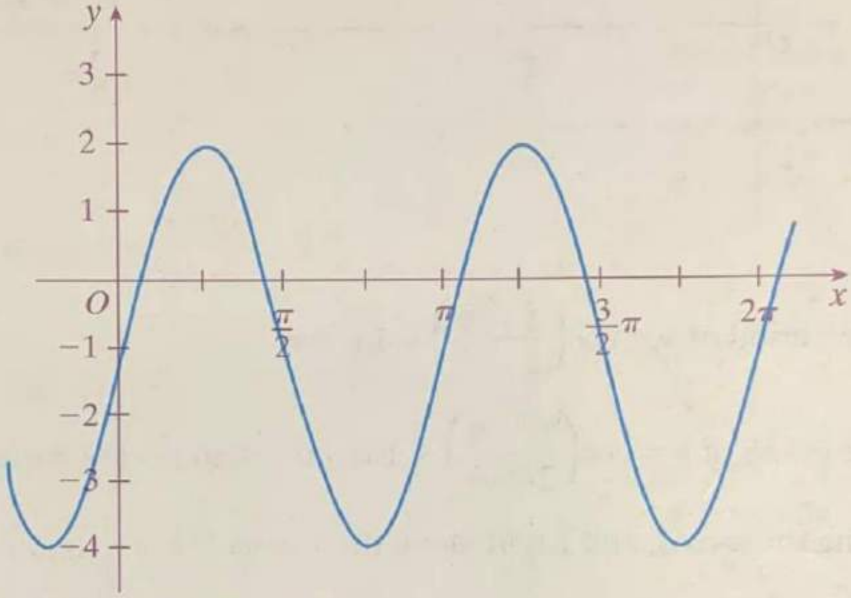
M 90 a Graphs of Trigonometric Functions 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

In each of the following exercises, draw the graph of the given function and state the period and translation.

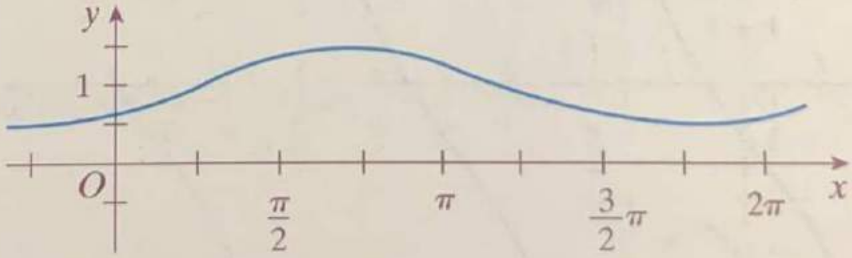
(1) $y = 3\sin 2x - 1$



The period of $y = 3\sin 2x - 1$ is π.

The graph of $y = 3\sin 2x - 1$ is a translation of $y = 3\sin 2x$, -1 unit along the y-axis.

(2) $y = \frac{1}{2}\sin\left(x - \frac{\pi}{4}\right) + 1$

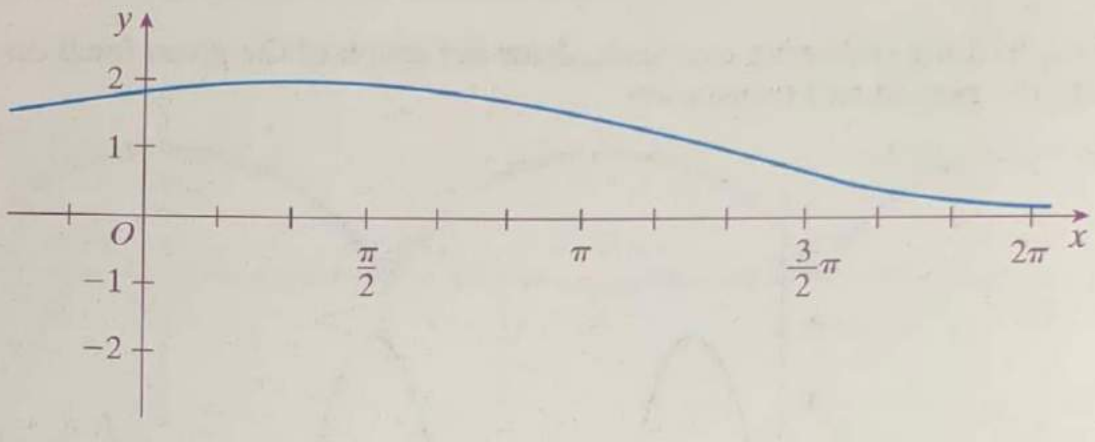


The period of $y = \frac{1}{2}\sin\left(x - \frac{\pi}{4}\right) + 1$ is 2π.

The graph of $y = \frac{1}{2}\sin\left(x - \frac{\pi}{4}\right) + 1$ is a translation of $y = \frac{1}{2}\sin x$, $\frac{\pi}{4}$ units along the x-axis, and 1 unit along the y-axis.

M 90 b

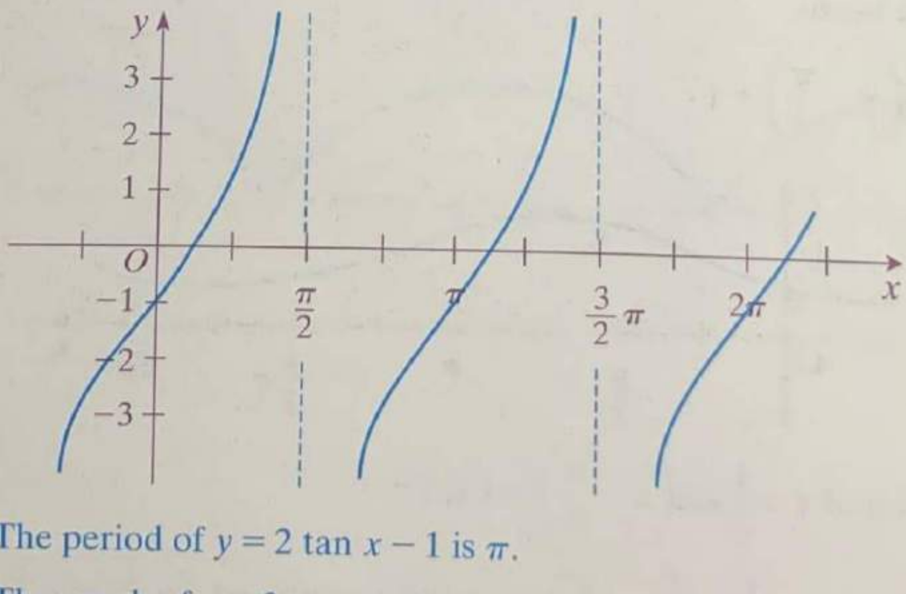
(3) $y = \cos\left(\frac{x}{2} - \frac{\pi}{6}\right) + 1$



The period of $y = \cos\left(\frac{x}{2} - \frac{\pi}{6}\right) + 1$ is 4π.

The graph of $y = \cos\left(\frac{x}{2} - \frac{\pi}{6}\right) + 1$ is a translation of $y = \cos\frac{x}{2}$, $\frac{\pi}{3}$ units along the x-axis, and 1 unit along the y-axis.

(4) $y = 2\tan x - 1$



The period of $y = 2\tan x - 1$ is π.

The graph of $y = 2\tan x - 1$ is a translation of $y = 2\tan x$, -1 unit along the y-axis.

KUMON

M 91 a Trigonometric Inequalities

Time : to : Date Name

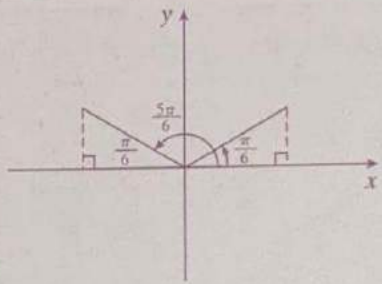
100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Given that $0 \leq x < 2\pi$, solve each of the following inequalities as shown in the example.

Ex. $\sin x > \dfrac{1}{2}$

[Sol] $\sin x = \dfrac{1}{2}$

when $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

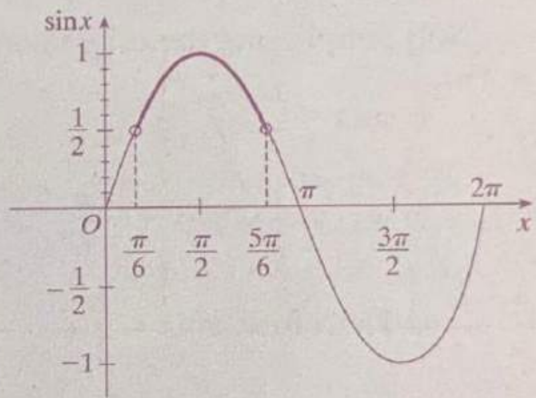


Note: The graph shows all the possible angles, x, in the given domain, at which $\sin x = \dfrac{1}{2}$.

Therefore,

$\sin x > \dfrac{1}{2}$

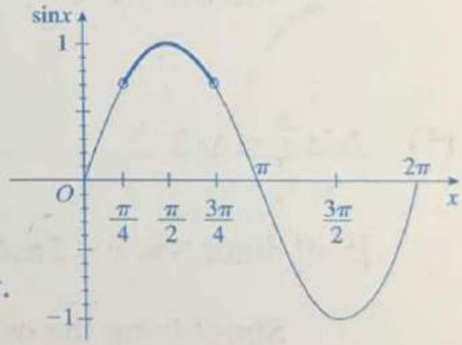
when $\dfrac{\pi}{6} < x < \dfrac{5\pi}{6}$



(1) $\sin x > \dfrac{\sqrt{2}}{2}$

[Sol] $\sin x = \dfrac{\sqrt{2}}{2}$ when $x = \dfrac{\pi}{4}, \dfrac{3}{4}\pi$

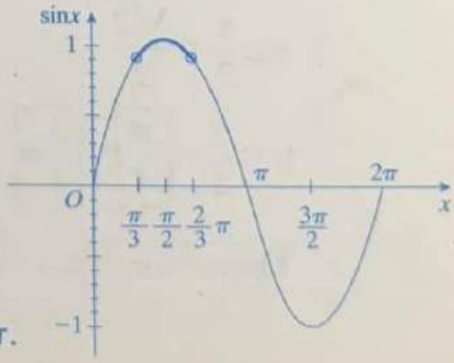
Therefore, $\sin x > \dfrac{\sqrt{2}}{2}$ when $\dfrac{\pi}{4} < x < \dfrac{3}{4}\pi$.



(2) $\sin x > \dfrac{\sqrt{3}}{2}$

[Sol] $\sin x = \dfrac{\sqrt{3}}{2}$ when $x = \dfrac{\pi}{3}, \dfrac{2}{3}\pi$

Therefore, $\sin x > \dfrac{\sqrt{3}}{2}$ when $\dfrac{\pi}{3} < x < \dfrac{2}{3}\pi$.



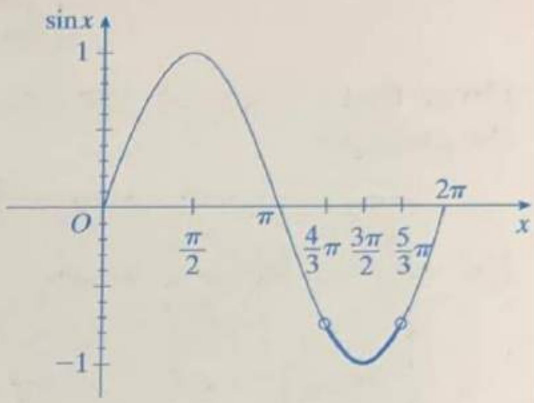
M 91 b

(3) $\sin x < -\frac{\sqrt{3}}{2}$

[Sol] $\sin x = -\frac{\sqrt{3}}{2}$ when $x = \frac{4}{3}\pi, \frac{5}{3}\pi$

Therefore, $\sin x < -\frac{\sqrt{3}}{2}$

when $\frac{4}{3}\pi < x < \frac{5}{3}\pi$



(4) $2\sin x < 1$

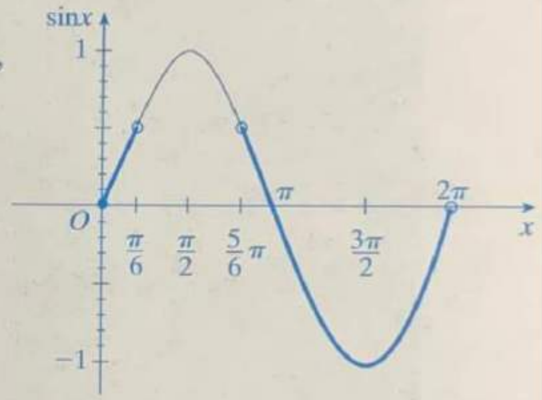
[Sol] Simplifying the original inequality,

$\sin x < \frac{1}{2}$

$\sin x = \frac{1}{2}$ when $x = \frac{\pi}{6}, \frac{5}{6}\pi$

Therefore, $\sin x < \frac{1}{2}$

when $0 \leq x < \frac{\pi}{6},\ \frac{5}{6}\pi < x < 2\pi$



(5) $2\sin\frac{x}{2} < \sqrt{2}$

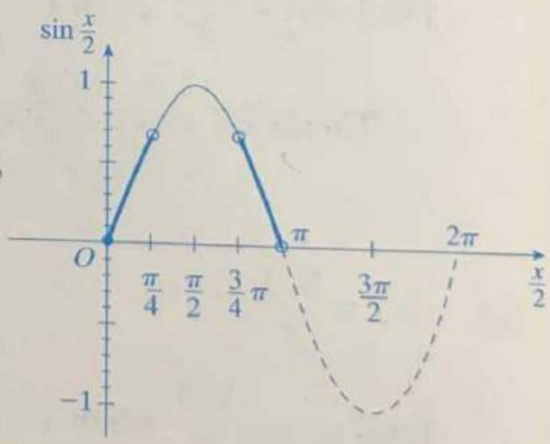
[Sol] Since $0 \leq x < 2\pi$, $0 \leq \frac{x}{2} < \boxed{\pi}$

Simplifying the original inequality,

$\sin\frac{x}{2} < \frac{\sqrt{2}}{2}$

$0 \leq \frac{x}{2} < \boxed{\frac{\pi}{4}},\ \boxed{\frac{3}{4}\pi} < \frac{x}{2} < \pi$

Therefore, $0 \leq x < \boxed{\frac{\pi}{2}},\ \boxed{\frac{3}{2}\pi} < x < 2\pi$.



Trigonometric Inequalities

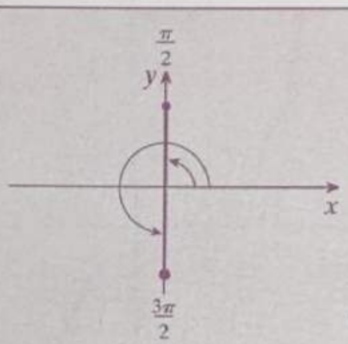
Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Given that $0 \leq x < 2\pi$, solve each of the following inequalities as shown in the example.

Ex. $\cos x > 0$

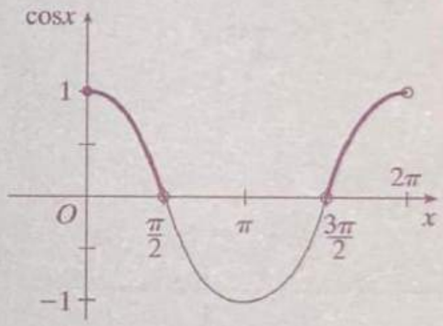
[Sol]



Note: The graph shows all the possible angles, x, in the given domain, at which $\cos x = 0$.

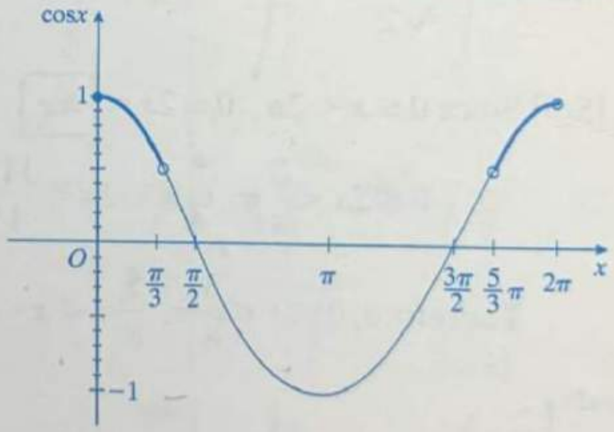
$\cos x > 0$

when $0 \leq x < \frac{\pi}{2}$, $\frac{3\pi}{2} < x < 2\pi$



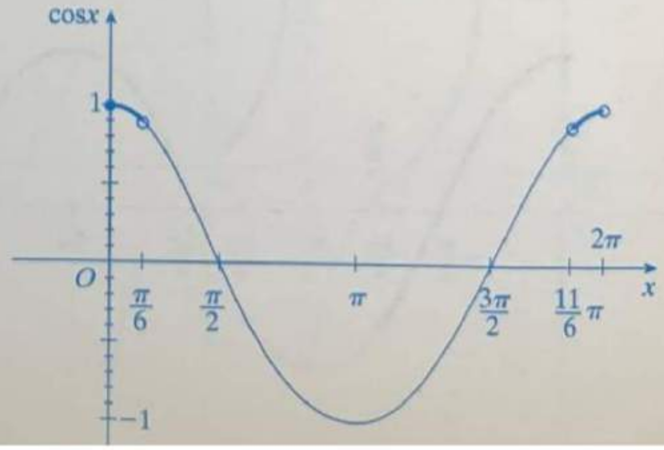
(1) $\cos x > \frac{1}{2}$

$0 \leq x < \frac{\pi}{3}$, $\frac{5}{3}\pi < x < 2\pi$

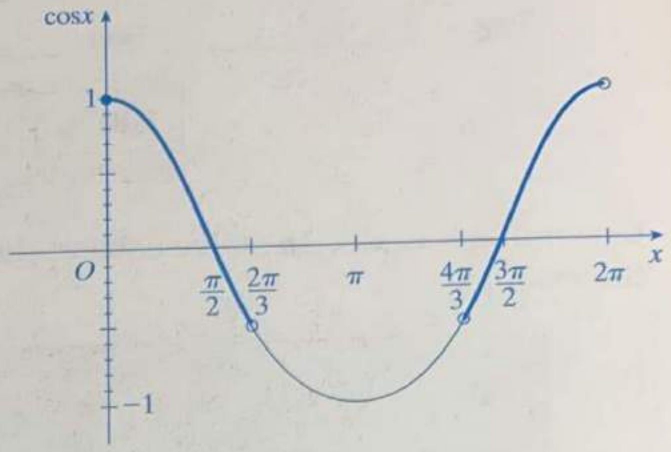


(2) $\cos x > \frac{\sqrt{3}}{2}$

$0 \leq x < \frac{\pi}{6}$, $\frac{11}{6}\pi < x < 2\pi$



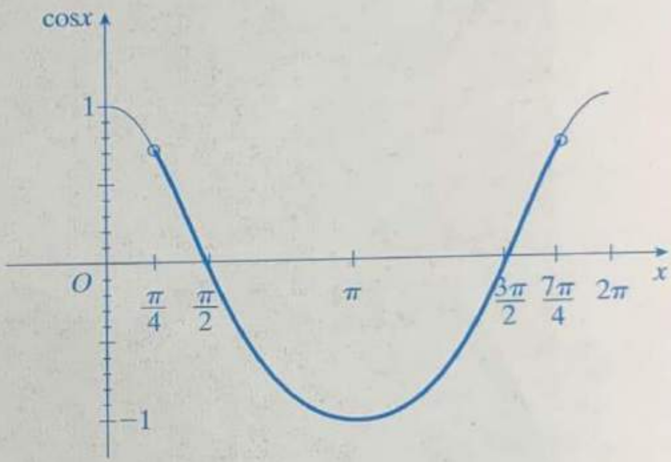
M 92 b



(3) $2\cos x > -1$

[Sol] $\cos x > -\frac{1}{2}$

$$0 \le x < \frac{2}{3}\pi,\ \frac{4}{3}\pi < x < 2\pi$$



(4) $\cos x < \frac{1}{\sqrt{2}}$

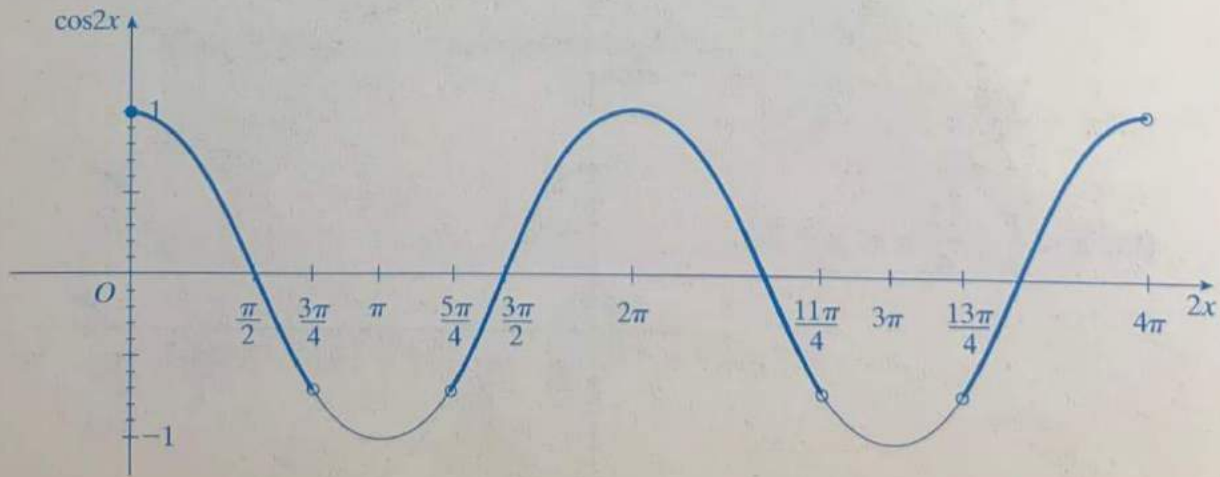
$$\frac{\pi}{4} < x < \frac{7}{4}\pi$$

(5) $\cos 2x > -\frac{1}{\sqrt{2}}$

[Sol] Since $0 \le x < 2\pi,\ 0 \le 2x < \boxed{4\pi}$

$$0 \le 2x < \frac{3}{4}\pi,\ \frac{5}{4}\pi < 2x < \frac{11}{4}\pi,\ \frac{13}{4}\pi < 2x < 4\pi$$

Therefore, $0 \le x < \frac{3}{8}\pi,\ \frac{5}{8}\pi < x < \frac{11}{8}\pi,\ \frac{13}{8}\pi < x < 2\pi$



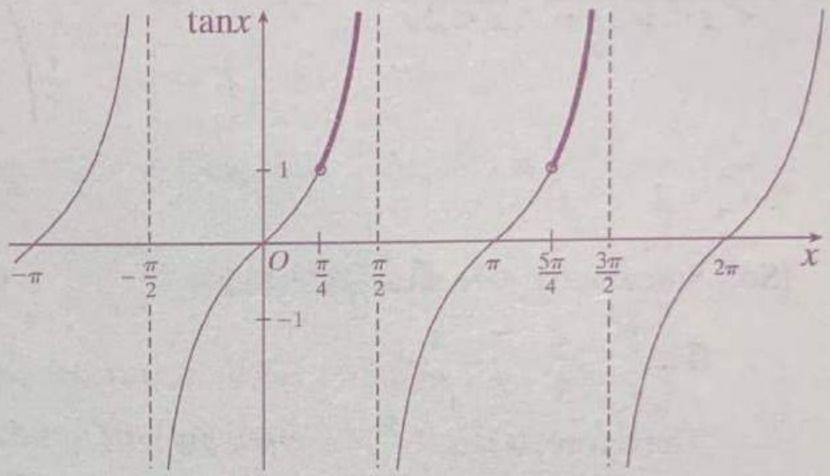
M 93 a Trigonometric Inequalities

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Given that $0 \leq x < 2\pi$, solve each of the following inequalities.

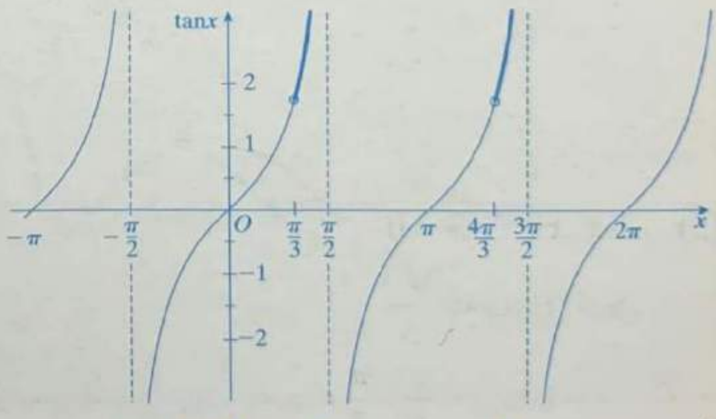
Ex. $\tan x > 1$



$\tan x > 1$

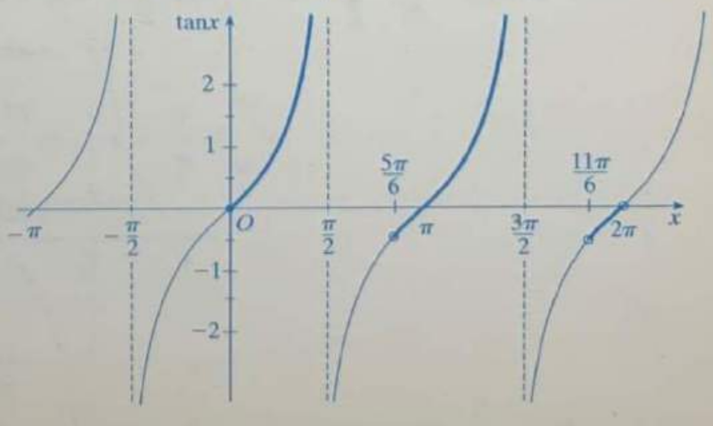
when $\frac{\pi}{4} < x < \frac{\pi}{2}$, $\frac{5}{4}\pi < x < \frac{3}{2}\pi$

(1) $\tan x > \sqrt{3}$



$\frac{\pi}{3} < x < \frac{\pi}{2}$, $\frac{4}{3}\pi < x < \frac{3}{2}\pi$

(2) $\tan x > -\frac{\sqrt{3}}{3}$

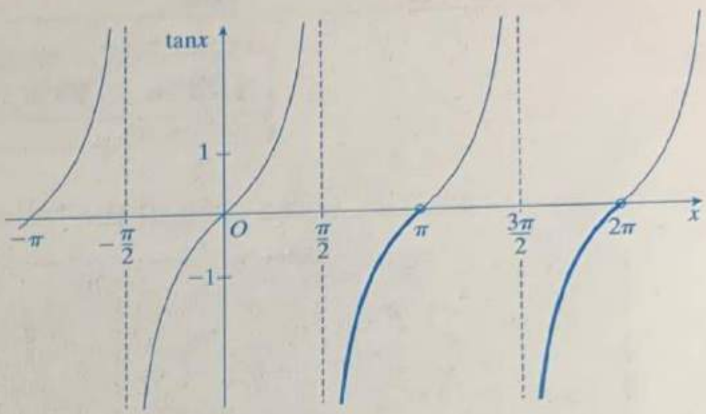


$0 \leq x < \frac{\pi}{2}$, $\frac{5}{6}\pi < x < \frac{3}{2}\pi$,

$\frac{11}{6}\pi < x < 2\pi$

M 93 b

(3) $\tan x < 0$



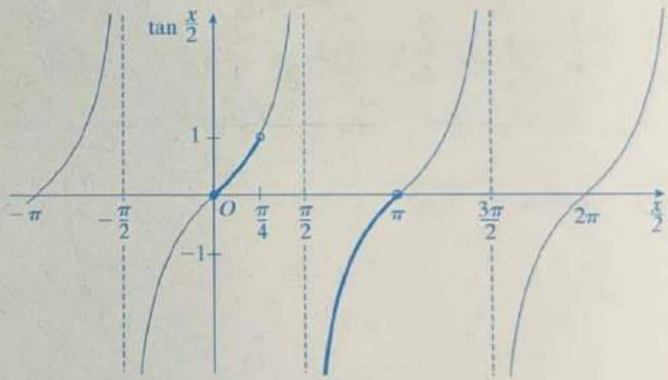
$\frac{\pi}{2} < x < \pi,\ \frac{3}{2}\pi < x < 2\pi$

(4) $\tan\frac{x}{2} < 1$

[Sol] Since $0 \le x < 2\pi,\ 0 \le \frac{x}{2} < \pi$

$0 \le \frac{x}{2} < \frac{\pi}{4},\ \frac{\pi}{2} < \frac{x}{2} < \pi$

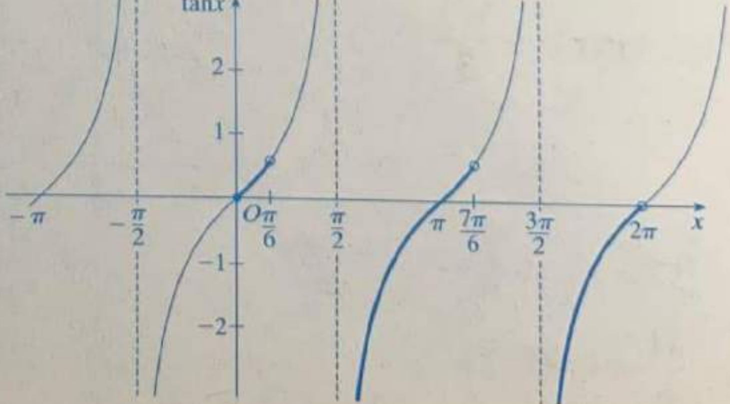
Therefore, $0 \le x < \frac{\pi}{2},\ \pi < x < 2\pi$



(5) $3\tan x - \sqrt{3} < 0$

[Sol] $\tan x < \frac{\sqrt{3}}{3}$

$0 \le x < \frac{\pi}{6},\ \frac{\pi}{2} < x < \frac{7}{6}\pi,\ \frac{3}{2}\pi < x < 2\pi$



M 94 a Trigonometric Inequalities

Time : to : Date Name

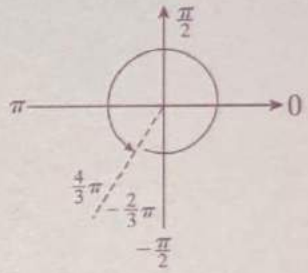
100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1~

Given that $-\pi < x < \pi$, solve each of the following inequalities as shown in the example.

Ex. $\sin\left(x+\dfrac{\pi}{3}\right) \geq -\dfrac{1}{2}$

[Sol] Since $-\pi < x < \pi$, Add $\dfrac{\pi}{3}$ to each term of the given domain.

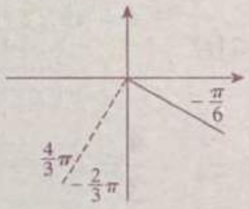
$-\dfrac{2}{3}\pi < x+\dfrac{\pi}{3} < \dfrac{4}{3}\pi$

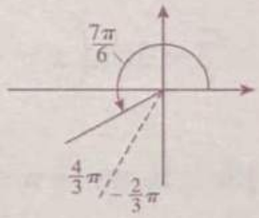


In this domain, determine all the possible angles at which the sin function equals $-\dfrac{1}{2}$.

The angles are:

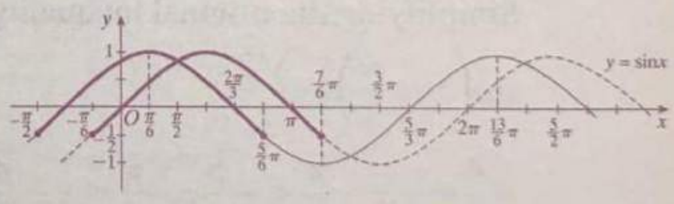
$-\dfrac{\pi}{6}, \dfrac{7}{6}\pi$





Solving the original inequality,

$-\dfrac{\pi}{6} \leq x+\dfrac{\pi}{3} \leq \dfrac{7}{6}\pi$



Therefore, $-\dfrac{\pi}{2} \leq x \leq \dfrac{5}{6}\pi$

(1) $\sin\left(x+\dfrac{\pi}{6}\right) \geq \dfrac{1}{2}$

[Sol] Since $-\pi < x < \pi$, $-\dfrac{5}{6}\pi < x+\dfrac{\pi}{6} < \dfrac{7}{6}\pi$

Solving the original inequality,

$\dfrac{\pi}{6} \leq x+\dfrac{\pi}{6} \leq \dfrac{5}{6}\pi$

Therefore, $0 \leq x \leq \dfrac{2}{3}\pi$

(2) $\sin\left(x-\frac{\pi}{6}\right)<\frac{\sqrt{3}}{2}$

[Sol] Since $-\pi<x<\pi$, $-\frac{7}{6}\pi<x-\frac{\pi}{6}<\frac{5}{6}\pi$

Solving the original inequality,

$-\frac{7}{6}\pi<x-\frac{\pi}{6}<\frac{\pi}{3}$, $\frac{2}{3}\pi<x-\frac{\pi}{6}<\frac{5}{6}\pi$

Therefore, $-\pi<x<\frac{\pi}{2}$, $\frac{5}{6}\pi<x<\pi$

(3) $\sqrt{2}-2\sin\left(x-\frac{\pi}{3}\right)<0$

[Sol] Since $-\pi<x<\pi$, $-\frac{4}{3}\pi<x-\frac{\pi}{3}<\frac{2}{3}\pi$

Simplifying the original inequality,

$\sin\left(x-\frac{\pi}{3}\right)>\frac{\sqrt{2}}{2}$

$-\frac{4}{3}\pi<x-\frac{\pi}{3}<-\frac{5}{4}\pi$, $\frac{\pi}{4}<x-\frac{\pi}{3}<\frac{2}{3}\pi$

Therefore, $-\pi<x<-\frac{11}{12}\pi$, $\frac{7}{12}\pi<x<\pi$

KUMON

M 95 a

Trigonometric Inequalities

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

Given that $-\pi < x < \pi$, solve each of the following inequalities.

(1) $\cos\left(2x - \frac{\pi}{6}\right) > \frac{\sqrt{3}}{2}$

[Sol] Since $-\pi < x < \pi$, $-\frac{13}{6}\pi < 2x - \frac{\pi}{6} < \frac{11}{6}\pi$

Solving the original inequality,

$-\frac{13}{6}\pi < 2x - \frac{\pi}{6} < -\frac{11}{6}\pi$, $-\frac{\pi}{6} < 2x - \frac{\pi}{6} < \frac{\pi}{6}$

Therefore, $-\pi < x < -\frac{5}{6}\pi$, $0 < x < \frac{\pi}{6}$

(2) $\cos\left(\frac{x}{2} + \pi\right) + \frac{1}{\sqrt{2}} > 0$

[Sol] Since $-\pi < x < \pi$, $\frac{\pi}{2} < \frac{x}{2} + \pi < \frac{3}{2}\pi$

Simplifying the original inequality,

$\cos\left(\frac{x}{2} + \pi\right) > -\frac{1}{\sqrt{2}}$

$\frac{\pi}{2} < \frac{x}{2} + \pi < \frac{3}{4}\pi$, $\frac{5}{4}\pi < \frac{x}{2} + \pi < \frac{3}{2}\pi$

Therefore, $-\pi < x < -\frac{\pi}{2}$, $\frac{\pi}{2} < x < \pi$

M 95 b

(3) $\tan\left(x-\frac{\pi}{4}\right)\leq 1$

[Sol] Since $-\pi<x<\pi$, $-\frac{5}{4}\pi<x-\frac{\pi}{4}<\frac{3}{4}\pi$

Solving the original inequality,

$-\frac{5}{4}\pi<x-\frac{\pi}{4}\leq-\frac{3}{4}\pi$, $-\frac{\pi}{2}<x-\frac{\pi}{4}\leq\frac{\pi}{4}$, $\frac{\pi}{2}<x-\frac{\pi}{4}<\frac{3}{4}\pi$

Therefore, $-\pi<x\leq-\frac{\pi}{2}$, $-\frac{\pi}{4}<x\leq\frac{\pi}{2}$, $\frac{3}{4}\pi<x<\pi$

(4) $\sqrt{3}\tan\left(\frac{x}{2}-\frac{\pi}{2}\right)+1\leq 0$

[Sol] Since $-\pi<x<\pi$, $-\pi<\frac{x}{2}-\frac{\pi}{2}<0$

Simplifying the original inequality,

$\tan\left(\frac{x}{2}-\frac{\pi}{2}\right)\leq-\frac{1}{\sqrt{3}}$

$-\frac{\pi}{2}<\frac{x}{2}-\frac{\pi}{2}\leq-\frac{\pi}{6}$

Therefore, $0<x\leq\frac{2}{3}\pi$

M 96 a Trigonometric Inequalities

Time : to : Date Name

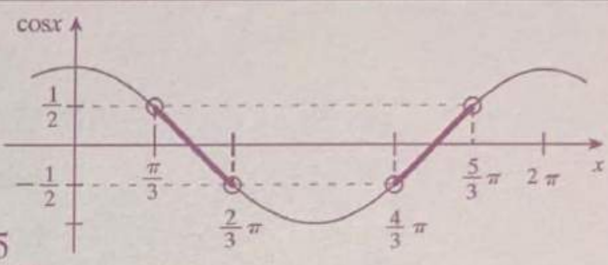
100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Given that $0 \leq x < 2\pi$, solve each of the following inequalities as shown in the example.

Ex. $-\dfrac{1}{2} < \cos x < \dfrac{1}{2}$

[Sol] From the graph,

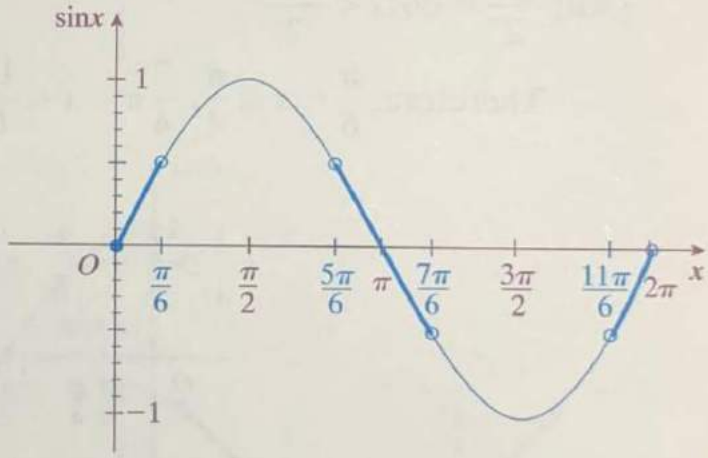
$\dfrac{\pi}{3} < x < \dfrac{2}{3}\pi,\ \dfrac{4}{3}\pi < x < \dfrac{5}{3}\pi$



(1) $-\dfrac{1}{2} < \sin x < \dfrac{1}{2}$

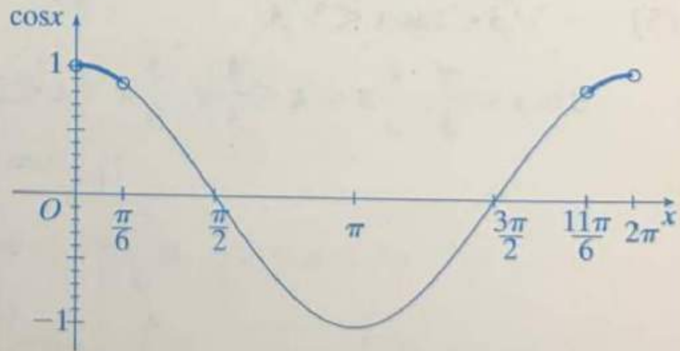
$0 \leq x < \dfrac{\pi}{6},\ \dfrac{5}{6}\pi < x < \dfrac{7}{6}\pi,$

$\dfrac{11}{6}\pi < x < 2\pi$



(2) $\dfrac{\sqrt{3}}{2} < \cos x < 1$

$0 < x < \dfrac{\pi}{6},\ \dfrac{11}{6}\pi < x < 2\pi$

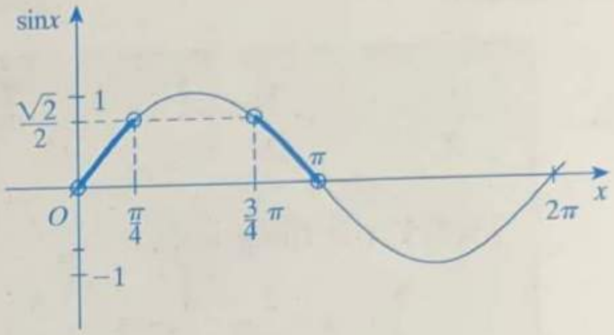


M 96 b

(3) $0 < 2\sin x < \sqrt{2}$

[Sol] $0 < \sin x < \boxed{\dfrac{\sqrt{2}}{2}}$

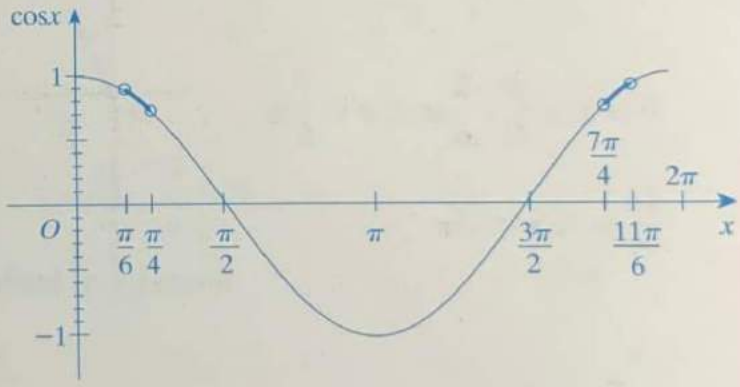
Therefore, $0 < x < \dfrac{\pi}{4}$, $\dfrac{3}{4}\pi < x < \pi$



(4) $\sqrt{2} < 2\cos x < \sqrt{3}$

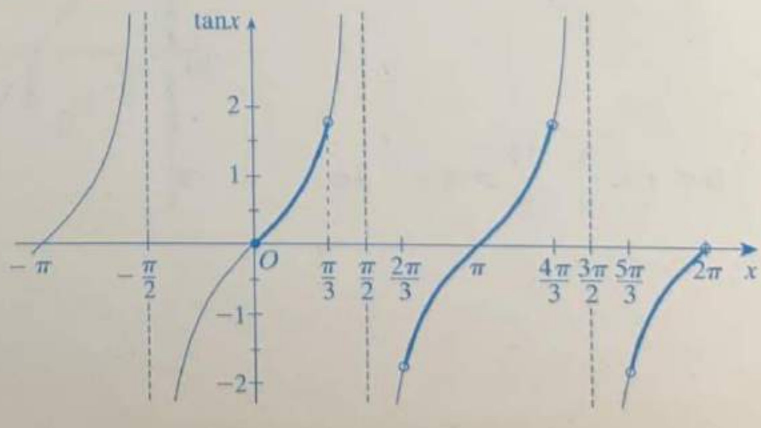
[Sol] $\dfrac{\sqrt{2}}{2} < \cos x < \dfrac{\sqrt{3}}{2}$

Therefore, $\dfrac{\pi}{6} < x < \dfrac{\pi}{4}$, $\dfrac{7}{4}\pi < x < \dfrac{11}{6}\pi$



(5) $-\sqrt{3} < \tan x < \sqrt{3}$

$0 \leq x < \dfrac{\pi}{3}$, $\dfrac{2}{3}\pi < x < \dfrac{4}{3}\pi$, $\dfrac{5}{3}\pi < x < 2\pi$



KUMON

M 97 a

Trigonometric Inequalities

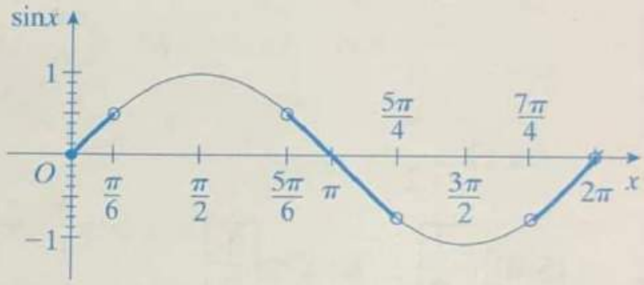
Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Given that $0 \leq x < 2\pi$, solve each of the following inequalities.

(1) $-\dfrac{\sqrt{2}}{2} < \sin x < \dfrac{1}{2}$

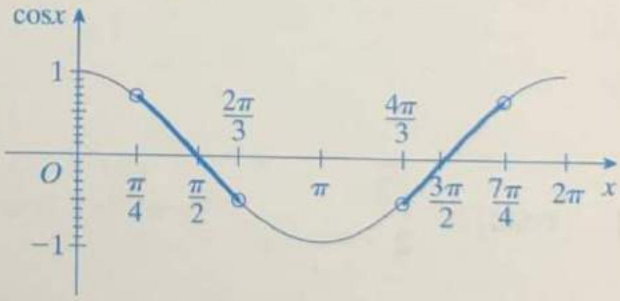
$0 \leq x < \dfrac{\pi}{6},\ \dfrac{5}{6}\pi < x < \dfrac{5}{4}\pi,\ \dfrac{7}{4}\pi < x < 2\pi$



(2) $-1 < 2\cos x < \sqrt{2}$

[Sol] $-\dfrac{1}{2} < \cos x < \dfrac{\sqrt{2}}{2}$

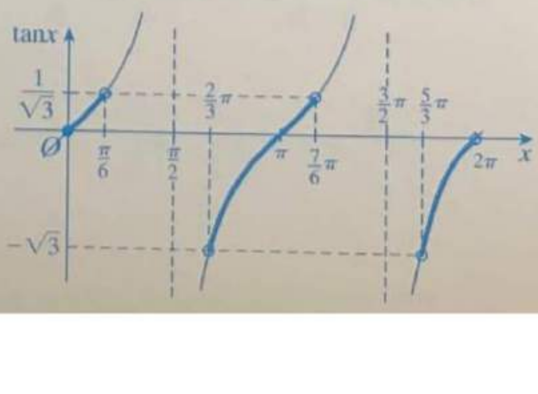
Therefore, $\dfrac{\pi}{4} < x < \dfrac{2}{3}\pi,\ \dfrac{4}{3}\pi < x < \dfrac{7}{4}\pi$



(3) $-2 < \sqrt{3}\tan x + 1 < 2$

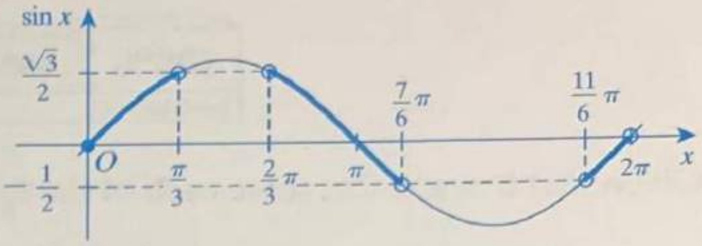
[Sol] $-\sqrt{3} < \tan x < \dfrac{1}{\sqrt{3}}$

Therefore, $0 \leq x < \dfrac{\pi}{6},\ \dfrac{2}{3}\pi < x < \dfrac{7}{6}\pi,\ \dfrac{5}{3}\pi < x < 2\pi$



M 97 b

(4) $0 < 2\sin x + 1 < 1 + \sqrt{3}$



[Sol] $-\frac{1}{2} < \sin x < \frac{\sqrt{3}}{2}$

Therefore,

$0 \leq x < \frac{\pi}{3},\ \frac{2}{3}\pi < x < \frac{7}{6}\pi,\ \frac{11}{6}\pi < x < 2\pi$

(5) $|\sin x| \leq \frac{1}{2}$

[Sol] $\boxed{-\frac{1}{2}} \leq \sin x \leq \boxed{\frac{1}{2}}$

Therefore,

$0 \leq x \leq \frac{\pi}{6},\ \frac{5}{6}\pi \leq x \leq \frac{7}{6}\pi,\ \frac{11}{6}\pi \leq x < 2\pi$

(6) $|\cos x| \geq \frac{\sqrt{3}}{2}$

[Sol] $\cos x \leq -\frac{\sqrt{3}}{2},\ \frac{\sqrt{3}}{2} \leq \cos x$

Therefore,

$0 \leq x \leq \frac{\pi}{6},\ \frac{5}{6}\pi \leq x \leq \frac{7}{6}\pi,\ \frac{11}{6}\pi \leq x < 2\pi$

M 98 a Trigonometric Inequalities

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Given that $0 \leq x < 2\pi$, solve each of the following inequalities as shown in the example.

Ex. $\cos x + 2\cos^2 x < 0$

Factor.

[Sol] $\cos x(1 + 2\cos x) < 0$

When $\cos x(1 + 2\cos x) = 0$, $\cos x = 0, -\frac{1}{2}$

$-\frac{1}{2} < \cos x < 0$

Therefore, $\frac{\pi}{2} < x < \frac{2}{3}\pi,\ \frac{4}{3}\pi < x < \frac{3}{2}\pi$

(1) $\sqrt{3}\sin x - 2\sin^2 x < 0$

[Sol] $2\sin^2 x - \sqrt{3}\sin x > 0$

$\sin x(2\sin x - \sqrt{3}) > 0$

$\sin x < 0,\ \sin x > \frac{\sqrt{3}}{2}$

Therefore, $\frac{\pi}{3} < x < \frac{2}{3}\pi,\ \pi < x < 2\pi$

(2) $\sqrt{3}\tan^2 x + 3\tan x > 0$

[Sol] $\tan x(\sqrt{3}\tan x + 3) > 0$

$\tan x < -\sqrt{3},\ 0 < \tan x$

Therefore,

$0 < x < \frac{\pi}{2},\ \frac{\pi}{2} < x < \frac{2}{3}\pi,\ \pi < x < \frac{3}{2}\pi,\ \frac{3}{2}\pi < x < \frac{5}{3}\pi$

M 98 b

(3) $\tan^2 x - 3 > 0$

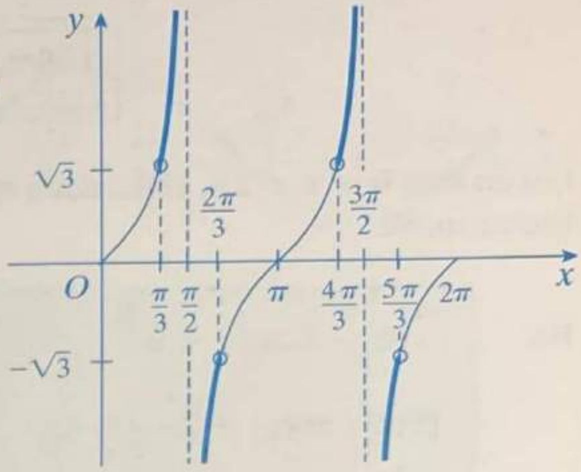
[Sol] $(\tan x + \sqrt{3})(\tan x - \sqrt{3}) > 0$

$\tan x < -\sqrt{3}$, $\tan x > \sqrt{3}$

Therefore,

$\frac{\pi}{3} < x < \frac{\pi}{2}$, $\frac{\pi}{2} < x < \frac{2}{3}\pi$,

$\frac{4}{3}\pi < x < \frac{3}{2}\pi$, $\frac{3}{2}\pi < x < \frac{5}{3}\pi$



(4) $2\sin^2 x \geq 1$

[Sol] $2\sin^2 x - 1 \geq 0$

$\sin^2 x - \frac{1}{2} \geq 0$

$\left(\sin x + \frac{1}{\sqrt{2}}\right)\left(\sin x - \frac{1}{\sqrt{2}}\right) \geq 0$

$\sin x \leq -\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}} \leq \sin x$

Therefore,

$\frac{\pi}{4} \leq x \leq \frac{3}{4}\pi$, $\frac{5}{4}\pi \leq x \leq \frac{7}{4}\pi$

(5) $2\sin^2 x > \sin x + 1$

[Sol] $2\sin^2 x - \sin x - 1 > 0$

$(2\sin x + 1)(\sin x - 1) > 0$

$\sin x < -\frac{1}{2}$ $\quad (\because \sin x - 1 \leq 0)$

Since $\sin x - 1 \leq 0$ is always true, we only need to examine the case $\sin x < -\frac{1}{2}$.

Therefore,

$\frac{7}{6}\pi < x < \frac{11}{6}\pi$

Trigonometric Inequalities

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. Given that $0 \leq x < 2\pi$, solve each of the following inequalities.

(1) $2\cos^2 x + \sin x - 1 > 0$

[Sol] $2(1 - \sin^2 x) + \sin x - 1 > 0$

$2\sin^2 x - \sin x - 1 < 0$

$(2\sin x + 1)(\sin x - 1) < 0$

$-\frac{1}{2} < \sin x < 1$

Therefore, $0 \leq x < \frac{\pi}{2},\ \frac{\pi}{2} < x < \frac{7}{6}\pi,\ \frac{11}{6}\pi < x < 2\pi$

(2) $2\sin^2 x < 3\cos x$

[Sol] $2(1 - \cos^2 x) < 3\cos x$

$2\cos^2 x + 3\cos x - 2 > 0$

$(2\cos x - 1)(\cos x + 2) > 0$

$\cos x > \frac{1}{2} \quad (\because \cos x + 2 > 0)$

Therefore, $0 \leq x < \frac{\pi}{3},\ \frac{5}{3}\pi < x < 2\pi$

(3) $2\cos^2 x + \sqrt{3}\sin x + 1 > 0$

[Sol] $2(1 - \sin^2 x) + \sqrt{3}\sin x + 1 > 0$

$2\sin^2 x - \sqrt{3}\sin x - 3 < 0$

$(2\sin x + \sqrt{3})(\sin x - \sqrt{3}) < 0$

$\sin x > -\frac{\sqrt{3}}{2} \quad (\because \sin x - \sqrt{3} < 0)$

Therefore, $0 \leq x < \frac{4}{3}\pi,\ \frac{5}{3}\pi < x < 2\pi$

M 99 b

2. Given that $-\pi < x < \pi$, solve each of the following inequalities.

(1) $1 + \sin x \geq 2\cos^2 x$

[Sol] $1 + \sin x \geq 2(1 - \sin^2 x)$

$2\sin^2 x + \sin x - 1 \geq 0$

$(2\sin x - 1)(\sin x + 1) \geq 0$

$\sin x \leq -1,\ \frac{1}{2} \leq \sin x$

Therefore, $x = -\frac{\pi}{2},\ \frac{\pi}{6} \leq x \leq \frac{5}{6}\pi$

(2) $\sin^2 x + 3\cos^2 x - 2\cos x - 1 \geq 0$

[Sol] $(1 - \cos^2 x) + 3\cos^2 x - 2\cos x - 1 \geq 0$

$2\cos^2 x - 2\cos x \geq 0$

$2\cos x(\cos x - 1) \geq 0$

$\cos x \leq 0,\ 1 \leq \cos x$

Therefore, $-\pi < x \leq -\frac{\pi}{2},\ x = 0,\ \frac{\pi}{2} \leq x < \pi$

Trigonometric Inequalities

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. Given $0 \leq x < 2\pi$, solve each of the following inequalities.

(1) $2\sin x > 1$

[Sol] $\sin x > \frac{1}{2}$

$\frac{\pi}{6} < x < \frac{5}{6}\pi$

(2) $\tan\frac{x}{2} > -1$

[Sol] Since $0 \leq x < 2\pi$, $0 \leq \frac{x}{2} < \pi$

Solving the original inequality,

$0 \leq \frac{x}{2} < \frac{\pi}{2}$, $\frac{3}{4}\pi < \frac{x}{2} < \pi$

Therefore, $0 \leq x < \pi$, $\frac{3}{2}\pi < x < 2\pi$

(3) $\frac{1}{2} < \cos x < 1$

$0 < x < \frac{\pi}{3}$, $\frac{5}{3}\pi < x < 2\pi$

(4) $2\cos^2 x < \cos x$

[Sol] $2\cos^2 x - \cos x < 0$

$\cos x(2\cos x - 1) < 0$

$0 < \cos x < \frac{1}{2}$

Therefore, $\frac{\pi}{3} < x < \frac{\pi}{2}$, $\frac{3}{2}\pi < x < \frac{5}{3}\pi$

2. Given $-\pi < x < \pi$, solve each of the following inequalities.

(1) $\sin\left(x-\frac{\pi}{3}\right)<\frac{1}{2}$

[Sol] Since $-\pi<x<\pi$, $-\frac{4}{3}\pi<x-\frac{\pi}{3}<\frac{2}{3}\pi$

Solving the original inequality,

$-\frac{7}{6}\pi<x-\frac{\pi}{3}<\frac{\pi}{6}$

Therefore, $-\frac{5}{6}\pi<x<\frac{\pi}{2}$

(2) $\cos\left(2x+\frac{\pi}{6}\right)>-\frac{\sqrt{2}}{2}$

[Sol] Since $-\pi<x<\pi$, $-\frac{11}{6}\pi<2x+\frac{\pi}{6}<\frac{13}{6}\pi$

Solving the original inequality,

$-\frac{11}{6}\pi<2x+\frac{\pi}{6}<-\frac{5}{4}\pi,\ -\frac{3}{4}\pi<2x+\frac{\pi}{6}<\frac{3}{4}\pi,\ \frac{5}{4}\pi<2x+\frac{\pi}{6}<\frac{13}{6}\pi$

Therefore,

$-\pi<x<-\frac{17}{24}\pi,\ -\frac{11}{24}\pi<x<\frac{7}{24}\pi,\ \frac{13}{24}\pi<x<\pi$

M 101 a

Maxima and Minima

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1~

Given that $0 \leq x < 2\pi$, find the maximum and minimum values of each of the following functions, and state the value of x at which each occurs.

Ex. $y = 3\sin x - 2$

[Sol] Since $0 \leq x < 2\pi$,

$$-1 \leq \sin x \leq 1$$

In each part, multiply by 3 and subtract 2.

$$\boxed{-5} \leq 3\sin x - 2 \leq \boxed{1}$$

When $\sin x = 1$, the function has a maximum value of $\boxed{1}$, at $x = \frac{\pi}{2}$.

When $\sin x = -1$, the function has a minimum value of $\boxed{-5}$, at $x = \boxed{\frac{3}{2}\pi}$.

Answers: -5, 1, 1, -5, $\frac{3}{2}\pi$

Complete and check your answers.

$y = 4\sin x - 1$

[Sol] Since $0 \leq x < 2\pi$,

$$\boxed{-1} \leq \sin x \leq \boxed{1}$$

$$\boxed{-5} \leq \boxed{4\sin x - 1} \leq 3$$

When $\sin x = 1$, the function has a maximum value of $\boxed{3}$, at $x = \boxed{\frac{\pi}{2}}$.

When $\sin x = -1$, the function has a minimum value of $\boxed{-5}$, at $x = \boxed{\frac{3}{2}\pi}$.

Answers: -1, 1, -5, $4\sin x - 1$, 3, $\frac{\pi}{2}$, -5, $\frac{3}{2}\pi$

M 101 b

(1) $y = 5\sin x - 3$

[Sol] Since $0 \leq x < 2\pi$,

$$-1 \leq \sin x \leq 1$$

$$-8 \leq 5\sin x - 3 \leq 2$$

When $\sin x = 1$, the function has a maximum value of 2, at $x = \frac{\pi}{2}$.

When $\sin x = -1$, the function has a minimum value of -8, at $x = \frac{3}{2}\pi$.

(2) $y = 2\sin x + 1$

[Sol] Since $0 \leq x < 2\pi$,

$$-1 \leq \sin x \leq 1$$

$$-1 \leq 2\sin x + 1 \leq 3$$

When $\sin x = 1$, the function has a maximum value of 3, at $x = \frac{\pi}{2}$.

When $\sin x = -1$, the function has a minimum value of -1, at $x = \frac{3}{2}\pi$.

Maxima and Minima

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1~

Given that $0 \leq x < 2\pi$, find the maximum and minimum values of each of the following functions, and state the value of x at which each occurs.

Ex. $y = 2\sin\left(x - \frac{\pi}{6}\right) + 1$

[Sol] Since $0 \leq x < 2\pi$,

Subtract $\frac{\pi}{6}$ from each part of the original domain.

$$\boxed{-\frac{\pi}{6}} \leq x - \frac{\pi}{6} < \boxed{\frac{11}{6}\pi}$$

Over this domain,

$$-1 \leq \sin\left(x - \frac{\pi}{6}\right) \leq 1$$

$$\boxed{-1} \leq 2\sin\left(x - \frac{\pi}{6}\right) + 1 \leq \boxed{3}$$

If $\sin\left(x - \frac{\pi}{6}\right) = 1$, then $\left(x - \frac{\pi}{6}\right) = \frac{\pi}{2}$, and $x = \boxed{\frac{2}{3}\pi}$.

If $\sin\left(x - \frac{\pi}{6}\right) = -1$, then $\left(x - \frac{\pi}{6}\right) = \boxed{\frac{3}{2}\pi}$, and $x = \boxed{\frac{5}{3}\pi}$.

Therefore,

When $\sin\left(x - \frac{\pi}{6}\right) = 1$, there is a maximum value of $\boxed{3}$, at $x = \boxed{\frac{2}{3}\pi}$.

When $\sin\left(x - \frac{\pi}{6}\right) = -1$, there is a minimum value of $\boxed{-1}$, at $x = \boxed{\frac{5}{3}\pi}$.

Answers: $-\frac{\pi}{6}$, $\frac{11}{6}\pi$, -1, 3, $\frac{2}{3}\pi$, $\frac{3}{2}\pi$, $\frac{5}{3}\pi$, 3, $\frac{2}{3}\pi$, -1, $\frac{5}{3}\pi$

M 102 b

(1) $y=3\sin\left(x-\frac{\pi}{3}\right)+2$

[Sol] Since $0 \leq x < 2\pi$,

$$-\frac{\pi}{3} \leq x - \frac{\pi}{3} < \frac{5}{3}\pi$$

Over this domain,

$$-1 \leq \sin\left(x-\frac{\pi}{3}\right) \leq 1$$

$$-1 \leq 3\sin\left(x-\frac{\pi}{3}\right)+2 \leq 5$$

When $\sin\left(x-\frac{\pi}{3}\right)=1$, there is a maximum value of 5, at $x=\frac{5}{6}\pi$.

When $\sin\left(x-\frac{\pi}{3}\right)=-1$, there is a minimum value of -1, at $x=\frac{11}{6}\pi$.

(2) $y=2\sin\left(x-\frac{\pi}{4}\right)-1$

[Sol] Since $0 \leq x < 2\pi$,

$$-\frac{\pi}{4} \leq x - \frac{\pi}{4} < \frac{7}{4}\pi$$

Over this domain,

$$-1 \leq \sin\left(x-\frac{\pi}{4}\right) \leq 1$$

$$-3 \leq 2\sin\left(x-\frac{\pi}{4}\right)-1 \leq 1$$

When $\sin\left(x-\frac{\pi}{4}\right)=1$, there is a maximum value of 1, at $x=\frac{3}{4}\pi$.

When $\sin\left(x-\frac{\pi}{4}\right)=-1$, there is a minimum value of -3, at $x=\frac{7}{4}\pi$.

M 103 a

Maxima and Minima

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Find the maximum and minimum values of each of the following functions in the given domain, and state the value of x at which each occurs.

Ex. $y = \tan x \quad \left(-\frac{\pi}{4} \leq x \leq \frac{\pi}{3}\right)$

[Sol] Since $\left(-\frac{\pi}{4} \leq x \leq \frac{\pi}{3}\right)$,

$-1 \leq \tan x \leq \sqrt{3}$

Take the tan of each part of the domain.

Therefore,

The maximum value is $\sqrt{3}$, at $x = \frac{\pi}{3}$.

The minimum value is -1, at $x = -\frac{\pi}{4}$.

(1) $y = 2\cos x - 3 \quad \left(\frac{\pi}{3} \leq x \leq \frac{7}{6}\pi\right)$

[Sol] From the given domain, $-1 \leq \cos x \leq \frac{1}{2}$

$-5 \leq 2\cos x - 3 \leq -2$

When $\cos x = \frac{1}{2}$, the function has a maximum value of -2, at $x = \frac{\pi}{3}$.

When $\cos x = -1$, the function has a minimum value of -5, at $x = \pi$.

(2) $y = 3\sin 2x + 1 \quad \left(-\frac{\pi}{3} \leq x \leq \frac{\pi}{6}\right)$

[Sol] From the given domain, $-\frac{2}{3}\pi \leq 2x \leq \frac{\pi}{3}$

$-1 \leq \sin 2x \leq \frac{\sqrt{3}}{2}$

$-2 \leq 3\sin 2x + 1 \leq \frac{2+3\sqrt{3}}{2}$

When $\sin 2x = \frac{\sqrt{3}}{2}$, the function has a maximum value of $\frac{2+3\sqrt{3}}{2}$, at $x = \frac{\pi}{6}$.

When $\sin 2x = -1$, the function has a minimum value of -2, at $x = -\frac{\pi}{4}$.

M 103 b

(3) $y=\cos\left(2x-\dfrac{\pi}{3}\right)$ $\left(\dfrac{\pi}{4}\leq x\leq\dfrac{3}{4}\pi\right)$

[Sol] From the given domain, $\dfrac{\pi}{6}\leq 2x-\dfrac{\pi}{3}\leq\dfrac{7}{6}\pi$

$$-1\leq\cos\left(2x-\frac{\pi}{3}\right)\leq\frac{\sqrt{3}}{2}$$

When $\cos\left(2x-\dfrac{\pi}{3}\right)=\dfrac{\sqrt{3}}{2}$, the function has a maximum value of $\dfrac{\sqrt{3}}{2}$, at $x=\dfrac{\pi}{4}$.

When $\cos\left(2x-\dfrac{\pi}{3}\right)=-1$, the function has a minimum value of -1, at $x=\dfrac{2}{3}\pi$.

(4) $y=2\tan\left(2x+\dfrac{\pi}{3}\right)-1$ $\left(-\dfrac{\pi}{12}\leq x\leq\dfrac{\pi}{12}\right)$

[Sol] From the given domain, $\dfrac{\pi}{6}\leq 2x+\dfrac{\pi}{3}\leq\dfrac{\pi}{2}$

$$\frac{\sqrt{3}}{3}\leq\tan\left(2x+\frac{\pi}{3}\right)$$

$$\frac{2\sqrt{3}-3}{3}\leq 2\tan\left(2x+\frac{\pi}{3}\right)-1$$

When $\tan\left(2x+\dfrac{\pi}{3}\right)=\dfrac{\sqrt{3}}{3}$, the function has a minimum value of $\dfrac{2\sqrt{3}-3}{3}$, at $x=-\dfrac{\pi}{12}$.
There is no maximum value.

(5) $y=\dfrac{4}{3}\cos\left(\dfrac{x}{2}+\dfrac{\pi}{6}\right)-\dfrac{2}{3}$ $(0\leq x<2\pi)$

[Sol] From the given domain, $\dfrac{\pi}{6}\leq\dfrac{x}{2}+\dfrac{\pi}{6}<\dfrac{7}{6}\pi$

$$-1\leq\cos\left(\frac{x}{2}+\frac{\pi}{6}\right)\leq\frac{\sqrt{3}}{2}$$

$$-2\leq\frac{4}{3}\cos\left(\frac{x}{2}+\frac{\pi}{6}\right)-\frac{2}{3}\leq\frac{2\sqrt{3}-2}{3}$$

When $\cos\left(\dfrac{x}{2}+\dfrac{\pi}{6}\right)=\dfrac{\sqrt{3}}{2}$, the function has a maximum value of $\dfrac{2\sqrt{3}-2}{3}$, at $x=0$.

When $\cos\left(\dfrac{x}{2}+\dfrac{\pi}{6}\right)=-1$, the function has a minimum value of -2, at $x=\dfrac{5}{3}\pi$.

Maxima and Minima

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1~

Given that $0 \leq x < 2\pi$, find the maximum and minimum values of each of the following functions, and state the value of x at which each occurs.

Ex. $y = \sin^2 x + \sin x + 3$

[Sol] Letting $\sin x = X$, $-1 \leq X \leq 1$

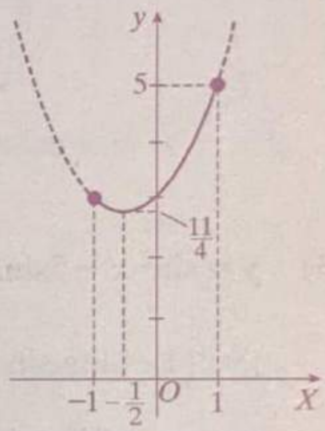
$$y = X^2 + X + 3 = \left(X + \boxed{\frac{1}{2}}\right)^2 + \boxed{\frac{11}{4}}$$

From the graph, the maximum is at $X = 1$,

and the minimum is at $X = \boxed{-\frac{1}{2}}$

When $X = \sin x = 1$, the function has a maximum value of $\boxed{5}$, at $x = \frac{\pi}{2}$.

When $X = \sin x = -\frac{1}{2}$, the function has a minimum value of $\boxed{\frac{11}{4}}$, at $x = \frac{7}{6}\pi$ or $\boxed{\frac{11}{6}\pi}$.



Answers: $\frac{1}{2}$, $\frac{11}{4}$, $-\frac{1}{2}$, 5, $\frac{11}{4}$, $\frac{11}{6}\pi$

(1) $y = \sin^2 x - \sin x + 1$

[Sol] Letting $\sin x = X$, $-1 \leq X \leq 1$

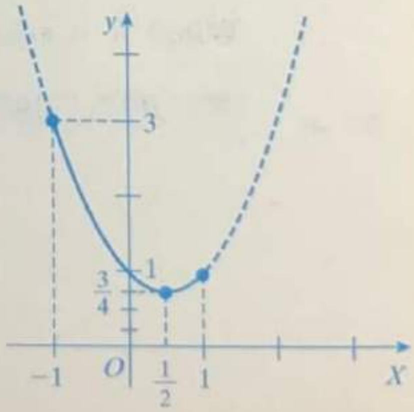
$$y = X^2 - X + 1 = \left(X - \frac{1}{2}\right)^2 + \frac{3}{4}$$

From the graph, the maximum is at $X = -1$,

and the minimum is at $X = \frac{1}{2}$

When $X = \sin x = -1$, the function has a maximum value of 3, at $x = \frac{3}{2}\pi$.

When $X = \sin x = \frac{1}{2}$, the function has a minimum value of $\frac{3}{4}$, at $x = \frac{\pi}{6}$ or $\frac{5}{6}\pi$.



(2) $y = \sin^2 x + 3\sin x - 1$

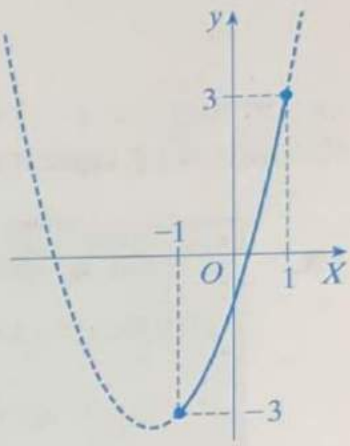
[Sol] Letting $\sin x = X$, $-1 \leq X \leq 1$

$$y = X^2 + 3X - 1 = \left(X + \frac{3}{2}\right)^2 - \frac{13}{4}$$

From the graph, the maximum is at $X = 1$,
and the minimum is at $X = -1$

When $X = \sin x = 1$, the function has a maximum value of 3, at $x = \dfrac{\pi}{2}$.

When $X = \sin x = -1$, the function has a minimum value of -3, at $x = \dfrac{3}{2}\pi$.



(3) $y = \sin^2 x - 5\sin x + 5$

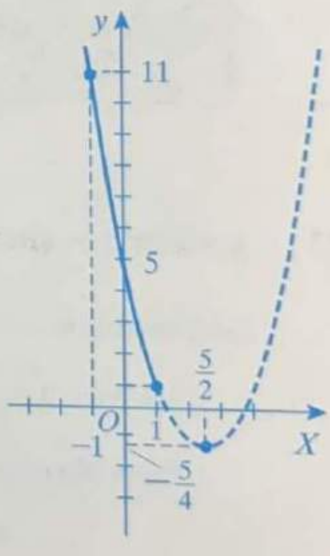
[Sol] Letting $\sin x = X$, $-1 \leq X \leq 1$

$$y = X^2 - 5X + 5 = \left(X - \frac{5}{2}\right)^2 - \frac{5}{4}$$

From the graph, the maximum is at $X = -1$,
and the minimum is at $X = 1$

When $X = \sin x = -1$, the function has a maximum value of 11, at $x = \dfrac{3}{2}\pi$.

When $X = \sin x = 1$, the function has a minimum value of 1, at $x = \dfrac{\pi}{2}$.



M 105 a

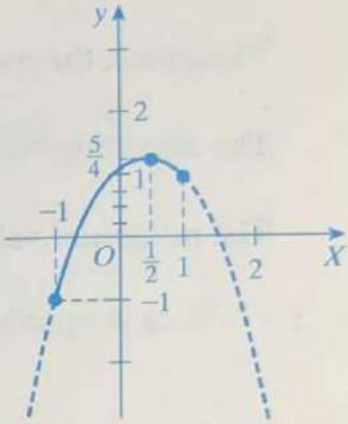
Maxima and Minima

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

Given that $0 \leq x < 2\pi$, find the maximum and minimum values of each of the following functions, and state the value of x at which each occurs.

(1) $y = \sin x + \cos^2 x$



[Sol] $y = \sin x + (1 - \sin^2 x)$

$= -\sin^2 x + \sin x + 1$

Letting $\sin x = X$, $-1 \leq X \leq 1$

$y = -X^2 + X + 1$

$= -\left(X - \frac{1}{2}\right)^2 + \frac{5}{4}$

Therefore, the maximum is at $X = \frac{1}{2}$, and the minimum is at $X = -1$.

The maximum value is $\frac{5}{4}$, at $x = \frac{\pi}{6}$ or $\frac{5}{6}\pi$.

The minimum value is -1, at $x = \frac{3}{2}\pi$.

(2) $y = \cos^2 x + \sqrt{3}\sin x$

[Sol] $y = (1 - \sin^2 x) + \sqrt{3}\sin x = -\sin^2 x + \sqrt{3}\sin x + 1$

Letting $\sin x = X$, $-1 \leq X \leq 1$

$y = -X^2 + \sqrt{3}X + 1 = -\left(X - \frac{\sqrt{3}}{2}\right)^2 + \frac{7}{4}$

Therefore, the maximum is at $X = \frac{\sqrt{3}}{2}$, and the minimum is at $X = -1$.

The maximum value is $\frac{7}{4}$, at $x = \frac{\pi}{3}$ or $\frac{2}{3}\pi$.

The minimum value is $-\sqrt{3}$, at $x = \frac{3}{2}\pi$.

(3) $y = 4(\cos x - \sin^2 x) - 1$

[Sol] $y = 4[\cos x - (1 - \cos^2 x)] - 1 = 4\cos^2 x + 4\cos x - 5$

Letting $\cos x = X, \quad -1 \leq X \leq 1$

$y = 4X^2 + 4X - 5 = 4\left(X + \frac{1}{2}\right)^2 - 6$

Therefore, the maximum is at $X = 1$, and the minimum is at $X = -\frac{1}{2}$.

The maximum value is 3, at $x = 0$.

The minimum value is -6, at $x = \frac{2}{3}\pi$ or $\frac{4}{3}\pi$.

(4) $y = 2 + \sin x - \cos^2 x$

[Sol] $y = 2 + \sin x - (1 - \sin^2 x) = \sin^2 x + \sin x + 1$

Letting $\sin x = X, \quad -1 \leq X \leq 1$

$y = X^2 + X + 1 = \left(X + \frac{1}{2}\right)^2 + \frac{3}{4}$

Therefore, the maximum is at $X = 1$, and the minimum is at $X = -\frac{1}{2}$.

The maximum value is 3, at $x = \frac{\pi}{2}$.

The minimum value is $\frac{3}{4}$, at $x = \frac{7}{6}\pi$ or $\frac{11}{6}\pi$.

M 106 a Maxima and Minima

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

Given that $0 \leq x < 2\pi$, find the maximum and minimum values of each of the following functions, and state the value of x at which each occurs.

(1) $y = \sin^2 x + 2\cos x + 1$

[Sol] $y = (1 - \cos^2 x) + 2\cos x + 1 = -\cos^2 x + 2\cos x + 2$

Letting $\cos x = X$, $-1 \leq X \leq 1$

$y = -X^2 + 2X + 2 = -(X-1)^2 + 3$

Therefore, the maximum is at $X = 1$, and the minimum is at $X = -1$.

The maximum value is 3, at $x = 0$.

The minimum value is -1, at $x = \pi$.

(2) $y = \cos^2 x - \sin x$

[Sol] $y = (1 - \sin^2 x) - \sin x = -\sin^2 x - \sin x + 1$

Letting $\sin x = X$, $-1 \leq X \leq 1$

$y = -X^2 - X + 1 = -\left(X + \frac{1}{2}\right)^2 + \frac{5}{4}$

Therefore, the maximum is at $X = -\frac{1}{2}$, and the minimum is at $X = 1$.

The maximum value is $\frac{5}{4}$, at $x = \frac{7}{6}\pi$ or $\frac{11}{6}\pi$.

The minimum value is -1, at $x = \frac{\pi}{2}$.

(3) $y = 2\sin^2 x + \sqrt{3}\sin x + \cos^2 x + 1$

[Sol] $y = 2\sin^2 x + \sqrt{3}\sin x + (1 - \sin^2 x) + 1 = \sin^2 x + \sqrt{3}\sin x + 2$

Letting $\sin x = X, \quad -1 \leq X \leq 1$

$$y = X^2 + \sqrt{3}X + 2 = \left(X + \frac{\sqrt{3}}{2}\right)^2 + \frac{5}{4}$$

Therefore, the maximum is at $X = 1$, and the minimum is at $X = -\frac{\sqrt{3}}{2}$.

The maximum value is $3 + \sqrt{3}$, at $x = \frac{\pi}{2}$.

The minimum value is $\frac{5}{4}$, at $x = \frac{4}{3}\pi$ or $\frac{5}{3}\pi$.

(4) $y = \cos^2 x + \sin x - 2$

[Sol] $y = (1 - \sin^2 x) + \sin x - 2 = -\sin^2 x + \sin x - 1$

Letting $\sin x = X, \quad -1 \leq X \leq 1$

$$y = -X^2 + X - 1 = -\left(X - \frac{1}{2}\right)^2 - \frac{3}{4}$$

Therefore, the maximum is at $X = \frac{1}{2}$, and the minimum is at $X = -1$.

The maximum value is $-\frac{3}{4}$, at $x = \frac{\pi}{6}$ or $\frac{5}{6}\pi$.

The minimum value is -3, at $x = \frac{3}{2}\pi$.

Maxima and Minima

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

Given that $0 \leq x < 2\pi$, find the maximum and minimum values of each of the following functions, and then state the range of the function.

(1) $y = 3\sin x - 2\cos^2 x$

[Sol] $y = 3\sin x - 2(\boxed{1-\sin^2 x})$

$= 2\sin^2 x + 3\boxed{\sin x} - \boxed{2}$

Letting $\sin x = t$, $-1 \leq t \leq 1$

$y = 2t^2 + \boxed{3t} - \boxed{2} = 2\left(t + \boxed{\frac{3}{4}}\right)^2 - \boxed{\frac{25}{8}}$

The maximum value is $\boxed{3}$, at $t = \boxed{1}$.

The minimum value is $\boxed{-\frac{25}{8}}$, at $t = \boxed{-\frac{3}{4}}$.

Therefore, the range is $\boxed{-\frac{25}{8}} \leq y \leq \boxed{3}$.

(2) $y = 2\cos x(\cos x - 3\tan x) - 2$

[Sol] $y = 2\cos^2 x - 6\sin x - 2$

$= 2(1 - \sin^2 x) - 6\sin x - 2$

$= 2 - 2\sin^2 x - 6\sin x - 2$

$= -2(\sin^2 x + 3\sin x)$

Letting $\sin x = t$, $-1 \leq t \leq 1$

$y = -2(t^2 + 3t) = -2\left(t + \frac{3}{2}\right)^2 + \frac{9}{2}$

The maximum value is 4, at $t = -1$.

The minimum value is -8, at $t = 1$.

Therefore, the range is $-8 \leq y \leq 4$.

M 107 b

(3) $y = 4 - \sin^2 x - 2\sin\left(\dfrac{\pi}{2} - x\right)$ **Hint**

[Sol] $y = 4 - (1 - \cos^2 x) - 2\cos x$

$= 4 - 1 + \cos^2 x - 2\cos x$

$= \cos^2 x - 2\cos x + 3$

Letting $\cos x = t$, $\quad -1 \le t \le 1$

$y = t^2 - 2t + 3 = (t-1)^2 + 2$

The maximum value is 6, at $t = -1$.

The minimum value is 2, at $t = 1$.

Therefore, the range is $2 \le y \le 6$.

(4) $y = \cos^2 x + \cos\left(\dfrac{\pi}{2} - x\right) + 1$

[Sol] $y = \cos^2 x + \sin x + 1$

$= -\sin^2 x + \sin x + 2$

Letting $\sin x = t$, $\quad -1 \le t \le 1$

$y = -t^2 + t + 2 = -\left(t - \dfrac{1}{2}\right)^2 + \dfrac{9}{4}$

The maximum value is $\dfrac{9}{4}$, at $t = \dfrac{1}{2}$.

The minimum value is 0, at $t = -1$.

Therefore, the range is $0 \le y \le \dfrac{9}{4}$.

Hint

First, express the function in terms of $\cos x$.

M 108 a

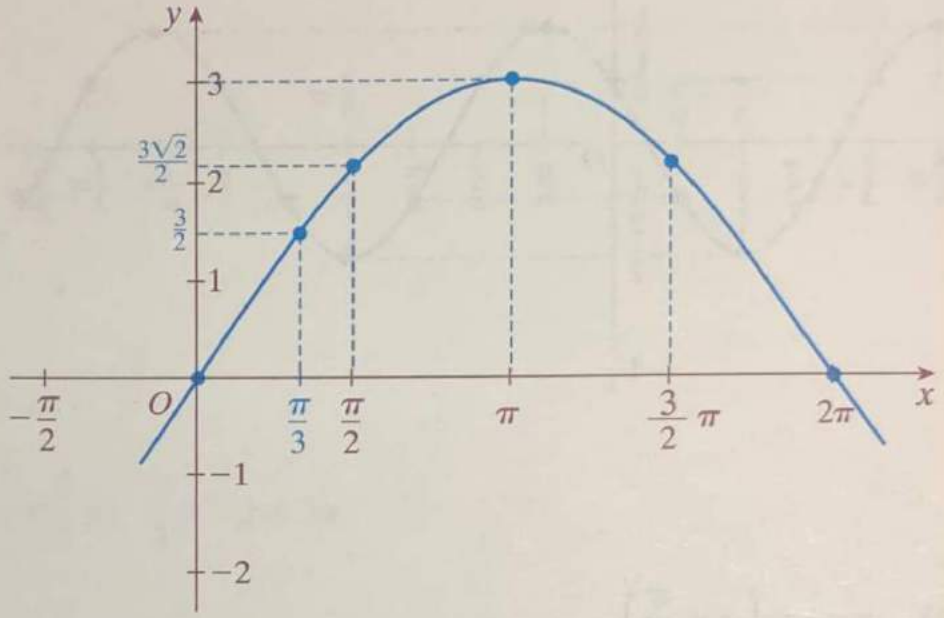
Maxima and Minima

Time : to : Date Name

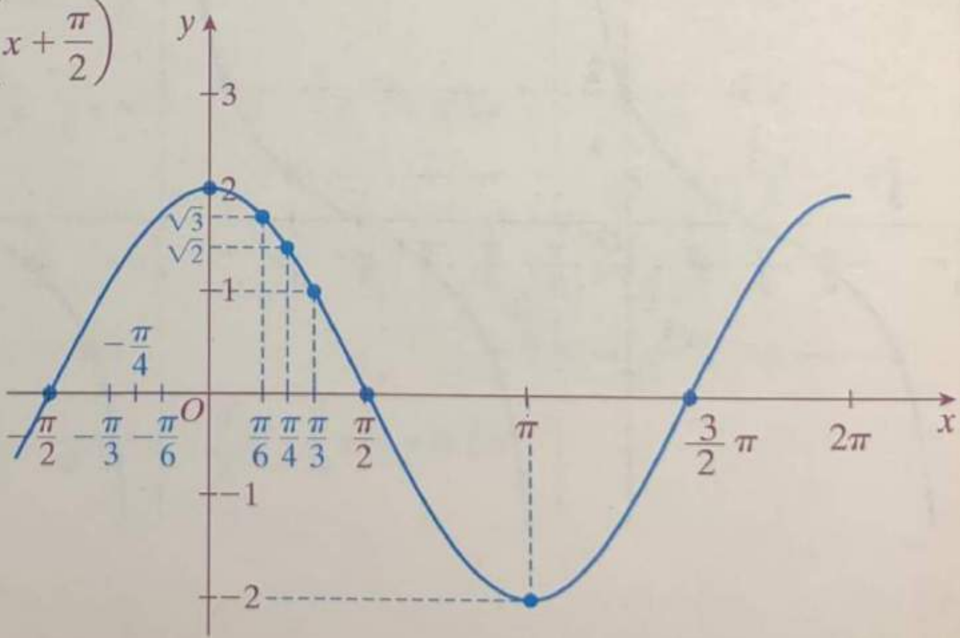
100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

Draw the graph of each of the following trigonometric functions.

(1) $y = 3\sin\frac{x}{2}$

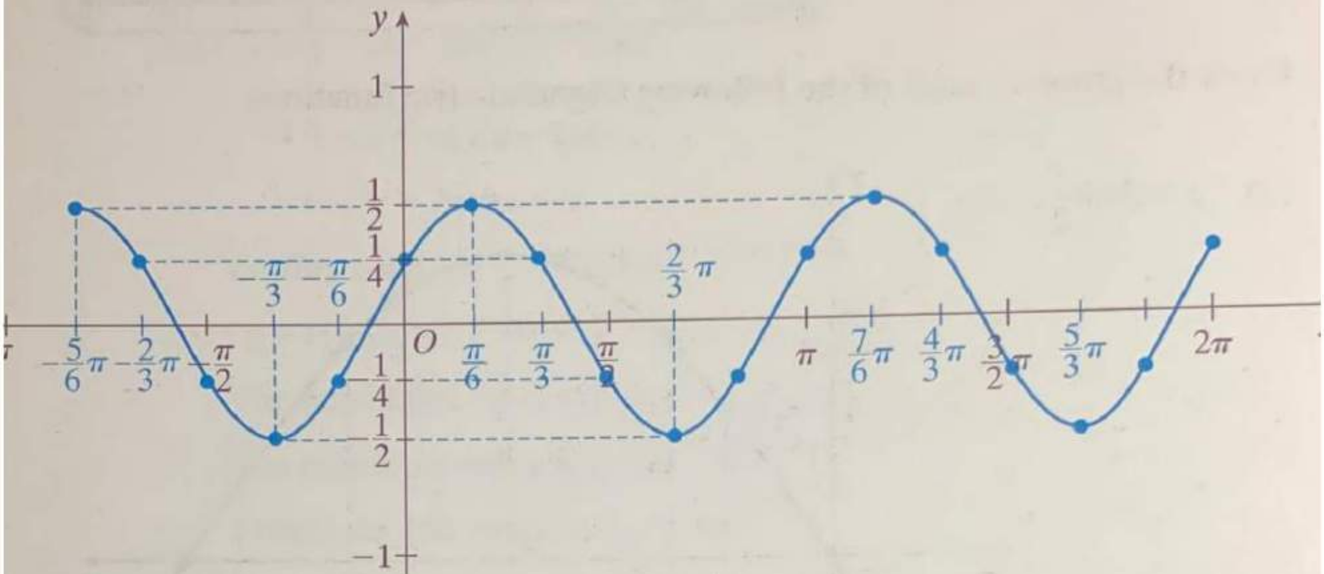


(2) $y = 2\sin\left(x + \frac{\pi}{2}\right)$

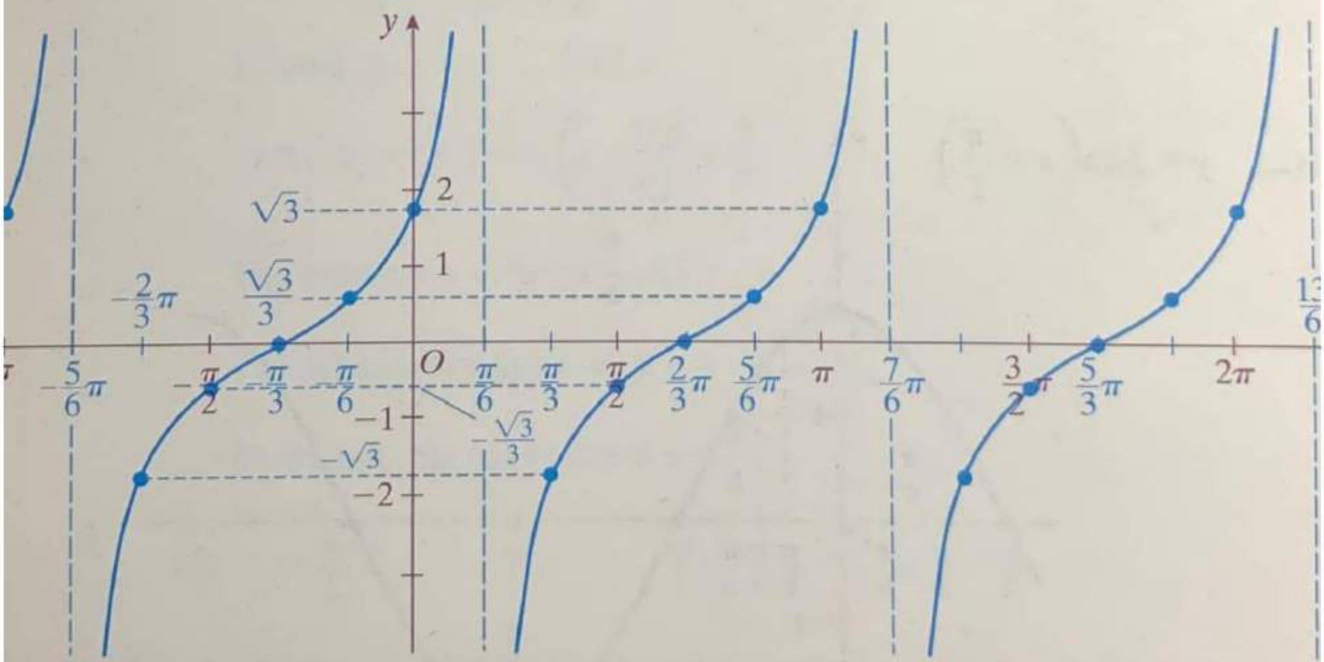


M 108 b

(3) $y=\frac{1}{2}\cos\left(2x-\frac{\pi}{3}\right)$



(4) $y=\tan\left(x+\frac{\pi}{3}\right)$



KUMON

M 109 a

Maxima and Minima

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

1. Given $0 \leq x \leq 2\pi$, solve each of the following inequalities.

(1) $\sin x > 0$

$0 < x < \pi$

(2) $\sqrt{2}\cos x > -1$

[Sol] $\cos x > -\dfrac{\sqrt{2}}{2}$

$0 \leq x < \dfrac{3}{4}\pi, \quad \dfrac{5}{4}\pi < x \leq 2\pi$

(3) $-\dfrac{\sqrt{3}}{2} < \sin x < \dfrac{1}{2}$

$0 \leq x < \dfrac{\pi}{6}, \quad \dfrac{5}{6}\pi < x < \dfrac{4}{3}\pi, \quad \dfrac{5}{3}\pi < x \leq 2\pi$

(4) $-1 < \tan x < \sqrt{3}$

$0 \leq x < \dfrac{\pi}{3}, \quad \dfrac{3}{4}\pi < x < \dfrac{4}{3}\pi, \quad \dfrac{7}{4}\pi < x \leq 2\pi$

M 109 b

2. Given $-\pi \leq x \leq \pi$, solve each of the following inequalities.

(1) $\cos\left(2x - \dfrac{\pi}{3}\right) > \dfrac{1}{2}$

[Sol] Since $-\pi \leq x \leq \pi$,

$$-2\pi \leq 2x \leq 2\pi$$

$$-\frac{7}{3}\pi \leq 2x - \frac{\pi}{3} \leq \frac{5}{3}\pi$$

Solving the inequality over this domain,

$$-\frac{7}{3}\pi < 2x - \frac{\pi}{3} < -\frac{5}{3}\pi, \quad -\frac{\pi}{3} < 2x - \frac{\pi}{3} < \frac{\pi}{3}$$

Therefore, $-\pi < x < -\dfrac{2}{3}\pi, \quad 0 < x < \dfrac{\pi}{3}$

(2) $\tan^2 x - 1 < 0$

[Sol] $(\tan x + 1)(\tan x - 1) < 0$

$$-1 < \tan x < 1$$

$$-\pi \leq x < -\frac{3}{4}\pi, \quad -\frac{\pi}{4} < x < \frac{\pi}{4}, \quad \frac{3}{4}\pi < x \leq \pi$$

(3) $3\sin x < 2\cos^2 x - 3$

[Sol] $3\sin x - 2\cos^2 x + 3 < 0$

$$3\sin x - 2(1 - \sin^2 x) + 3 < 0$$

$$2\sin^2 x + 3\sin x + 1 < 0$$

$$(2\sin x + 1)(\sin x + 1) < 0$$

$$-1 < \sin x < -\frac{1}{2}$$

Therefore, $-\dfrac{5}{6}\pi < x < -\dfrac{\pi}{2}, \quad -\dfrac{\pi}{2} < x < -\dfrac{\pi}{6}$

Maxima and Minima

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

1. Given that $0 \leq x < 2\pi$, find the maximum and minimum values of each of the following functions, and state the value of x at which each occurs.

(1) $y = 2\sin x + 1$

[Sol] Since $0 \leq x < 2\pi$,

$$-1 \leq \sin x \leq 1$$

$$-1 \leq 2\sin x + 1 \leq 3$$

When $\sin x = 1$, the function has a maximum value of 3, at $x = \frac{\pi}{2}$.

When $\sin x = -1$, the function has a minimum value of -1, at $x = \frac{3}{2}\pi$.

(2) $y = 3\sin\left(x - \frac{\pi}{6}\right) + 2$

[Sol] Since $0 \leq x < 2\pi$,

$$-\frac{\pi}{6} \leq x - \frac{\pi}{6} < \frac{11}{6}\pi$$

Over this domain,

$$-1 \leq \sin\left(x - \frac{\pi}{6}\right) \leq 1$$

$$-1 \leq 3\sin\left(x - \frac{\pi}{6}\right) + 2 \leq 5$$

When $\sin\left(x - \frac{\pi}{6}\right) = 1$, there is a maximum value of 5, at $x = \frac{2}{3}\pi$.

When $\sin\left(x - \frac{\pi}{6}\right) = -1$, there is a minimum value of -1, at $x = \frac{5}{3}\pi$.

2. Given that $0 \le x < 2\pi$, find the maximum and minimum values of each of the following functions, and then state the range of the function.

(1) $y = \sin^2 x + \sin x + 5$

[Sol] Letting $\sin x = t$, $-1 \le t \le 1$

$$y = t^2 + t + 5 = \left(t + \frac{1}{2}\right)^2 + \frac{19}{4}$$

The maximum value is 7, at $t = 1$.

The minimum value is $\frac{19}{4}$, at $t = -\frac{1}{2}$.

Therefore, the range is $\frac{19}{4} \le y \le 7$.

(2) $y = \cos^2 x + 2\cos\left(\frac{\pi}{2} - x\right) - 4$

[Sol] $y = \cos^2 x + 2\sin x - 4$

$= (1 - \sin^2 x) + 2\sin x - 4$

$= -\sin^2 x + 2\sin x - 3$

Letting $\sin x = t$, $-1 \le t \le 1$

$y = -t^2 + 2t - 3 = -(t-1)^2 - 2$

The maximum value is -2, at $t = 1$.

The minimum value is -6, at $t = -1$.

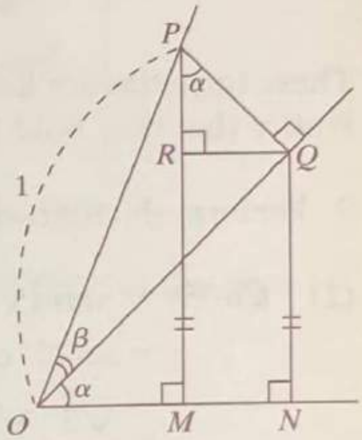
Therefore, the range is $-6 \le y \le -2$.

M 111 a Addition Theorem 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Having learned about trigonometric functions, we will now learn how to add different functions together.



Theorem $\boxed{\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta}$

1. Prove the Theorem.

Proof: Let $OP=1, 0<\alpha<\frac{\pi}{2}, 0<\beta<\frac{\pi}{2}$ and $0<\alpha+\beta<\frac{\pi}{2}$ in the diagram at right.

Then, the following relationships exist:

$$\sin\alpha=\frac{\boxed{QN}}{OQ} \quad \ldots ① \qquad \cos\alpha=\frac{\boxed{ON}}{OQ}=\frac{\boxed{PR}}{PQ} \quad \ldots ③$$

$$\sin\beta=\frac{\boxed{PQ}}{OP}=PQ \quad \ldots ② \qquad \cos\beta=\frac{\boxed{OQ}}{OP}=OQ \quad \ldots ④$$

Deriving new equations from the diagram,

$$\begin{aligned}\sin(\alpha+\beta)&=\frac{PM}{OP}=\frac{PM}{1}\\&=RM+PR\\&=\boxed{QN}+PR\\&=\left(\frac{OQ}{OQ}\right)QN+\left(\frac{PQ}{PQ}\right)PR\\&=\left(\frac{QN}{OQ}\right)OQ+\left(\frac{PR}{PQ}\right)PQ\\&=\boxed{\sin\alpha\cos\beta+\cos\alpha\sin\beta}\end{aligned}$$

See on the graph that: $\overline{RM}=\overline{QN}$

Using ①,②,③,④

In the same way:

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

Also: $\tan(\alpha+\beta)=\dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$

Dividing both the numerator and the denominator by $(\cos\alpha\cos\beta)$,

$$\tan(\alpha+\beta)=\frac{\boxed{\tan\alpha+\tan\beta}}{\boxed{1-\tan\alpha\tan\beta}}$$

M 111 b

Addition Theorem

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

These formulas are known as the **Addition Theorem**.
Notice that they hold for all values of α and β.

2. Perform the following calculations using the Addition Theorem.

(1) $$\sin 75^\circ = \sin(45^\circ + 30^\circ)$$
$$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

(2) $$\cos 75^\circ = \cos(45^\circ + 30^\circ)$$
$$= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

(3) $$\tan 75^\circ = \tan(45^\circ + 30^\circ)$$
$$= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} = \frac{1+\dfrac{\sqrt{3}}{3}}{1 - 1\cdot\dfrac{\sqrt{3}}{3}} = 2 + \sqrt{3}$$

(4) $$\cos 105^\circ = \cos(60^\circ + 45^\circ)$$
$$= \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ$$
$$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}-\sqrt{6}}{4}$$

KUMON

M 112 a

Addition Theorem 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Addition Theorem

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

1. Obtain the formulas for $(\alpha-\beta)$ by substituting $(-\beta)$ for β in the above Addition Theorem.

(1) $\sin(\alpha-\beta) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$

$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$

(2) $\cos(\alpha-\beta) = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta)$

$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$

(3) $\tan(\alpha-\beta) = \dfrac{\tan\alpha + \tan(-\beta)}{1 - \tan\alpha\tan(-\beta)}$

$= \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

Formulas

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

Notice that these subtraction formulas hold for all values of α and β.

2. Perform the following calculations using the subtraction formulas.

(1) $\sin 15^\circ = \sin(45^\circ - 30^\circ)$

$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

(2) $\cos 15^\circ = \cos(45^\circ - 30^\circ)$

$$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

(3) $\tan 15^\circ = \tan(45^\circ - 30^\circ)$

$$= \frac{\tan 45^\circ - \tan 30^\circ}{1+\tan 45^\circ \tan 30^\circ}$$
$$= \frac{1-\frac{\sqrt{3}}{3}}{1+1\cdot\frac{\sqrt{3}}{3}}$$
$$= 2-\sqrt{3}$$

Addition Theorem 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Perform the following calculations.

(1) $\sin 105^\circ = \sin(45^\circ + 60^\circ)$

$= \sin 45^\circ \cos 60^\circ + \cos 45^\circ \sin 60^\circ$

$= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

(2) $\cos 105^\circ = \cos(45^\circ + 60^\circ)$

$= \cos 45^\circ \cos 60^\circ - \sin 45^\circ \sin 60^\circ$

$= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}$

$= \frac{\sqrt{2}-\sqrt{6}}{4}$

(3) $\tan 105^\circ = \tan(45^\circ + 60^\circ)$

$= \frac{\tan 45^\circ + \tan 60^\circ}{1 - \tan 45^\circ \tan 60^\circ}$

$= \frac{1+\sqrt{3}}{1-1\cdot\sqrt{3}} = \frac{1+\sqrt{3}}{1-\sqrt{3}}$

$= \frac{4+2\sqrt{3}}{-2} = -2-\sqrt{3}$

M 113 b

(4) $\sin 15° = \sin(60° - 45°)$

$= \sin 60° \cos 45° - \cos 60° \sin 45°$

$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$

$= \frac{\sqrt{6} - \sqrt{2}}{4}$

(5) $\cos 15° = \cos(60° - 45°)$

$= \cos 60° \cos 45° + \sin 60° \sin 45°$

$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$

$= \frac{\sqrt{2} + \sqrt{6}}{4}$

(6) $\tan 15° = \tan(60° - 45°)$

$= \frac{\tan 60° - \tan 45°}{1 + \tan 60° \tan 45°}$

$= \frac{\sqrt{3} - 1}{1 + \sqrt{3}} = \frac{-4 + 2\sqrt{3}}{-2}$

$= 2 - \sqrt{3}$

M 114 a

Addition Theorem 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. Prove each of the following equalities using the Addition Theorem.

(1) $\sin 2\alpha = 2\sin\alpha\cos\alpha$

[Proof] $\sin 2\alpha = \sin(\alpha + \alpha)$

$= \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$

$= 2\sin\alpha\cos\alpha = \text{RHS}$

(2) $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$(= 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1)$

[Proof] $\cos 2\alpha = \cos(\alpha + \alpha)$

$= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$

$= \cos^2\alpha - \sin^2\alpha = \text{RHS}$

$$\left[\begin{array}{l} = 1 - \sin^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha \\ = \cos^2\alpha - (1 - \cos^2\alpha) = 2\cos^2\alpha - 1 \end{array}\right]$$

(3) $\tan 2\alpha = \dfrac{2\tan\alpha}{1 - \tan^2\alpha}$

[Proof] $\tan 2\alpha = \tan(\alpha + \alpha)$

$$= \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha\tan\alpha}$$

$$= \frac{2\tan\alpha}{1 - \tan^2\alpha} = \text{RHS}$$

Formulas

$$\begin{aligned}\sin 2\alpha &= 2\sin\alpha\cos\alpha \\ \cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\ &= 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1 \\ \tan 2\alpha &= \frac{2\tan\alpha}{1-\tan^2\alpha}\end{aligned}$$

These formulas are called the **double-angle formulas**.

2. Obtain the values of $\sin 2\alpha$ and $\cos 2\alpha$ when $\cos\alpha = \frac{4}{5}$.

$\left(0 < \alpha < \frac{\pi}{2}\right)$

[Sol] Since $0 < \alpha < \frac{\pi}{2}$, $\sin\alpha = \boxed{\frac{3}{5}}$.

$$\therefore\ \sin 2\alpha = 2\sin\alpha\cos\alpha = 2\cdot\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$$

3. Obtain the values of $\sin 2\alpha$, $\cos 2\alpha$ and $\tan 2\alpha$ when $\tan\alpha = -\frac{1}{3}$.

$\left(\frac{\pi}{2} < \alpha < \pi\right)$

[Sol] Since $\frac{\pi}{2} < \alpha < \pi$, $\sin\alpha = \frac{1}{\sqrt{10}}$ and $\cos\alpha = -\frac{3}{\sqrt{10}}$.

$$\therefore\ \sin 2\alpha = 2\sin\alpha\cos\alpha = 2\cdot\frac{1}{\sqrt{10}}\cdot\left(-\frac{3}{\sqrt{10}}\right) = -\frac{3}{5}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \frac{9}{10} - \frac{1}{10} = \frac{4}{5}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{-\frac{2}{3}}{1-\frac{1}{9}} = -\frac{3}{4}$$

Addition Theorem 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. Obtain the values of $\sin 2\alpha$ and $\cos 2\alpha$ when $\cos\alpha = \frac{12}{13}$.

$\left(0<\alpha<\frac{\pi}{2}\right)$

[Sol] Since $0<\alpha<\frac{\pi}{2}$, $\sin\alpha = \frac{5}{13}$.

$\therefore\ \sin 2\alpha = 2\sin\alpha\cos\alpha = 2\cdot\frac{5}{13}\cdot\frac{12}{13} = \frac{120}{169}$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \frac{144}{169} - \frac{25}{169} = \frac{119}{169}$

2. Obtain the values of $\sin 2\alpha$ and $\cos 2\alpha$ when $\sin\alpha = \frac{2}{5}$.

$\left(0<\alpha<\frac{\pi}{2}\right)$

[Sol] Since $0<\alpha<\frac{\pi}{2}$, $\cos\alpha = \frac{\sqrt{21}}{5}$.

$\therefore\ \sin 2\alpha = 2\sin\alpha\cos\alpha = 2\cdot\frac{2}{5}\cdot\frac{\sqrt{21}}{5} = \frac{4\sqrt{21}}{25}$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \frac{21}{25} - \frac{4}{25} = \frac{17}{25}$

3. Obtain the values of $\sin 2\alpha$ and $\cos 2\alpha$ when $\tan\alpha = \frac{4}{5}$.

$\left(0<\alpha<\frac{\pi}{2}\right)$

[Sol] Since $0<\alpha<\frac{\pi}{2}$, $\sin\alpha = \frac{4\sqrt{41}}{41}$ and $\cos\alpha = \frac{5\sqrt{41}}{41}$.

$\therefore\ \sin 2\alpha = 2\sin\alpha\cos\alpha = 2\cdot\frac{4\sqrt{41}}{41}\cdot\frac{5\sqrt{41}}{41} = \frac{40}{41}$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \frac{25}{41} - \frac{16}{41} = \frac{9}{41}$

4. Obtain the values of $\cos 2\alpha$ and $\tan 2\alpha$ when $\sin\alpha = -\dfrac{1}{\sqrt{3}}$.

$\left(\dfrac{3\pi}{2} < \alpha < 2\pi\right)$

[Sol] Since $\dfrac{3\pi}{2} < \alpha < 2\pi$, $\cos\alpha = \dfrac{\sqrt{6}}{3}$ and $\tan\alpha = -\dfrac{\sqrt{2}}{2}$.

$$\therefore \cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\cdot\left(-\dfrac{\sqrt{2}}{2}\right)}{1-\dfrac{1}{2}} = -2\sqrt{2}$$

5. Obtain the values of $\sin 2\alpha$, $\cos 2\alpha$ and $\tan 2\alpha$ when $\sin\alpha = -\dfrac{3}{\sqrt{13}}$.

$\left(\pi < \alpha < \dfrac{3\pi}{2}\right)$

[Sol] Since $\pi < \alpha < \dfrac{3\pi}{2}$, $\cos\alpha = -\dfrac{2}{\sqrt{13}}$ and $\tan\alpha = \dfrac{3}{2}$.

$$\therefore \sin 2\alpha = 2\sin\alpha\cos\alpha = 2\cdot\left(-\frac{3}{\sqrt{13}}\right)\cdot\left(-\frac{2}{\sqrt{13}}\right) = \frac{12}{13}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \frac{4}{13} - \frac{9}{13} = -\frac{5}{13}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\cdot\left(\dfrac{3}{2}\right)}{1-\dfrac{9}{4}} = -\frac{12}{5}$$

Addition Theorem 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. Prove each of the following formulas using the double-angle formulas.

(1) $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$

[Proof] $\sin 3\alpha = \sin(2\alpha + \alpha)$

$$= \sin 2\alpha \cos\alpha + \cos 2\alpha \sin\alpha$$
$$= 2\sin\alpha \cos\alpha \cos\alpha + (1 - 2\sin^2\alpha)\sin\alpha$$
$$= 2\sin\alpha(1 - \sin^2\alpha) + \sin\alpha - 2\sin^3\alpha$$
$$= 3\sin\alpha - 4\sin^3\alpha = \text{RHS}$$

(2) $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

[Proof] $\cos 3\alpha = \cos(2\alpha + \alpha)$

$$= \cos 2\alpha \cos\alpha - \sin 2\alpha \sin\alpha$$
$$= (2\cos^2\alpha - 1)\cos\alpha - 2\sin^2\alpha \cos\alpha$$
$$= 2\cos^3\alpha - \cos\alpha - 2(1 - \cos^2\alpha)\cos\alpha$$
$$= 4\cos^3\alpha - 3\cos\alpha = \text{RHS}$$

2. Obtain the values of $\sin 3\alpha$ and $\cos 3\alpha$ when $\sin\alpha = \frac{3}{5}$. $\left(0 < \alpha < \frac{\pi}{2}\right)$

[Sol] Since $0 < \alpha < \frac{\pi}{2}$, $\cos\alpha = \frac{4}{5}$.

$$\therefore \sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha = 3 \cdot \frac{3}{5} - 4\left(\frac{3}{5}\right)^3 = \frac{117}{125}$$

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha = 4\left(\frac{4}{5}\right)^3 - 3 \cdot \frac{4}{5} = -\frac{44}{125}$$

M 116 b

Formulas
$$\boxed{\begin{aligned}\sin 3\alpha &= 3\sin\alpha - 4\sin^3\alpha \\ \cos 3\alpha &= 4\cos^3\alpha - 3\cos\alpha\end{aligned}}$$

3. Obtain the values of $\sin 3\alpha$ and $\cos 3\alpha$ when $\cos\alpha = \frac{2}{3}$. $\left(\frac{3\pi}{2} < \alpha < 2\pi\right)$

[Sol] Since $\frac{3\pi}{2} < \alpha < 2\pi$, $\sin\alpha = -\frac{\sqrt{5}}{3}$.

$$\therefore\ \sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha = 3\cdot\left(-\frac{\sqrt{5}}{3}\right) - 4\left(-\frac{\sqrt{5}}{3}\right)^3 = -\frac{7\sqrt{5}}{27}$$

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha = 4\left(\frac{2}{3}\right)^3 - 3\left(\frac{2}{3}\right) = -\frac{22}{27}$$

4. Obtain the values of $\sin 3\alpha$ and $\cos 3\alpha$ when $\tan\alpha = \frac{\sqrt{7}}{3}$. $\left(\pi < \alpha < \frac{3\pi}{2}\right)$

[Sol] Since $\pi < \alpha < \frac{3\pi}{2}$, $\sin\alpha = -\frac{\sqrt{7}}{4}$ and $\cos\alpha = -\frac{3}{4}$.

$$\therefore\ \sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha = 3\cdot\left(-\frac{\sqrt{7}}{4}\right) - 4\left(-\frac{\sqrt{7}}{4}\right)^3 = -\frac{5\sqrt{7}}{16}$$

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha = 4\left(-\frac{3}{4}\right)^3 - 3\left(-\frac{3}{4}\right) = \frac{9}{16}$$

Addition Theorem 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. Prove each of the following equalities using the double-angle formulas.

(1) $\sin^2\dfrac{\alpha}{2}=\dfrac{1-\cos\alpha}{2}$

[Proof] $\cos\alpha=\cos\left(2\cdot\dfrac{\alpha}{2}\right)=1-2\sin^2\dfrac{\alpha}{2}$

$\therefore\ \sin^2\dfrac{\alpha}{2}=\dfrac{\boxed{1-\cos\alpha}}{2}$

(2) $\cos^2\dfrac{\alpha}{2}=\dfrac{1+\cos\alpha}{2}$

[Proof] $\cos\alpha=\cos\left(2\cdot\dfrac{\alpha}{2}\right)=\boxed{2\cos^2\dfrac{\alpha}{2}-1}$

$\therefore\ \cos^2\dfrac{\alpha}{2}=\boxed{\dfrac{1+\cos\alpha}{2}}$

(3) $\tan^2\dfrac{\alpha}{2}=\dfrac{1-\cos\alpha}{1+\cos\alpha}$

[Proof] $\tan^2\dfrac{\alpha}{2}=\dfrac{\sin^2\dfrac{\alpha}{2}}{\cos^2\dfrac{\alpha}{2}}$

$\therefore\ \tan^2\dfrac{\alpha}{2}=\dfrac{1-\cos\alpha}{1+\cos\alpha}$

Formulas

$$\sin^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{2}$$

$$\cos^2\frac{\alpha}{2}=\frac{1+\cos\alpha}{2}$$

$$\tan^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{1+\cos\alpha}$$

The above formulas are known as the **half-angle formulas.**

2. Evaluate each of the following using the half-angle formulas.

(1) $\sin 15°$

[Sol] $\sin^2 15° = \dfrac{1-\boxed{\cos 30°}}{2} = \dfrac{2-\sqrt{3}}{4}$

Since $\sin 15° > 0$,

$\sin 15° = \sqrt{\boxed{\dfrac{2-\sqrt{3}}{4}}} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

(2) $\tan 22.5°$

[Sol] $\tan^2 22.5° = \dfrac{1-\cos 45°}{1+\cos 45°} = \dfrac{2-\sqrt{2}}{2+\sqrt{2}} = 3-2\sqrt{2}$

Since $\tan 22.5° > 0$,

$\tan 22.5° = \sqrt{3-2\sqrt{2}} = \sqrt{2}-1$

(3) $\cos 105°$

[Sol] $\cos^2 105° = \dfrac{1+\cos 210°}{2} = \dfrac{1-\cos 30°}{2} = \dfrac{2-\sqrt{3}}{4}$

Since $\cos 105° < 0$,

$\cos 105° = -\sqrt{\dfrac{2-\sqrt{3}}{4}} = -\dfrac{\sqrt{6}-\sqrt{2}}{4}$

M 118 a

Addition Theorem 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

1. Evaluate each of the following trigonometric expressions.

(1) $\sin 22.5°$

[Sol] Since $\sin 22.5° > 0$,

$$\sin 22.5° = \sqrt{\frac{1-\cos 45°}{2}}$$

$$= \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

(2) $\cos 22.5°$

[Sol] Since $\cos 22.5° > 0$,

$$\cos 22.5° = \sqrt{\frac{1+\cos 45°}{2}}$$

$$= \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

(3) $\sin 11.25°$

[Sol] Since $\sin 11.25° > 0$,

$$\sin 11.25° = \sqrt{\frac{1-\cos 22.5°}{2}}$$

$$= \sqrt{\frac{1-\frac{\sqrt{2+\sqrt{2}}}{2}}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}$$

2. Evaluate $\sin\frac{\alpha}{2}$ and $\cos\frac{\alpha}{2}$ when $\sin\alpha = -\frac{4}{5}$. Pay close attention to the signs. $\left(\pi < \alpha < \frac{3}{2}\pi\right)$

[Sol] Since $\pi < \alpha < \frac{3}{2}\pi$, $\cos\alpha = -\frac{3}{5}$.

Since $\frac{\pi}{2} < \frac{\alpha}{2} < \frac{3}{4}\pi$, $\sin\frac{\alpha}{2} > 0$ and $\cos\frac{\alpha}{2} < 0$.

$$\therefore \sin\frac{\alpha}{2} = \sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}} = \sqrt{\frac{4}{5}} = \frac{2\sqrt{5}}{5}$$

$$\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\left(-\frac{3}{5}\right)}{2}} = -\sqrt{\frac{1}{5}} = -\frac{\sqrt{5}}{5}$$

3. Evaluate each of the following when $\cos\beta = \frac{5}{13}$. $\left(\frac{3}{2}\pi < \beta < 2\pi\right)$

(1) $\sin\frac{\beta}{2}$

[Sol] Since $\frac{3}{4}\pi < \frac{\beta}{2} < \pi$, $\sin\frac{\beta}{2} > 0$.

$$\therefore \sin\frac{\beta}{2} = \sqrt{\frac{1-\frac{5}{13}}{2}} = \sqrt{\frac{4}{13}} = \frac{2\sqrt{13}}{13}$$

(2) $\cos\frac{\beta}{2}$

[Sol] $\cos\frac{\beta}{2} < 0$

$$\therefore \cos\frac{\beta}{2} = -\sqrt{\frac{1+\frac{5}{13}}{2}} = -\sqrt{\frac{9}{13}} = -\frac{3\sqrt{13}}{13}$$

(3) $\tan\frac{\beta}{2}$

[Sol] $\tan\frac{\beta}{2} = \dfrac{\sin\frac{\beta}{2}}{\cos\frac{\beta}{2}} = -\frac{2}{3}$

Addition Theorem 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. Using the various formulas, express each of the following trigonometric functions in terms of t when $\tan\frac{\alpha}{2}=t$.

(1) $\tan\alpha$

[Sol] $\tan\alpha = \tan\left(2\cdot\frac{\alpha}{2}\right) = \dfrac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}$

$$=\frac{2t}{1-t^2}$$

(2) $\cos\alpha$ **Hint:** Use the half-angle formula for tan.

[Sol] $\tan^2\frac{\alpha}{2}=\dfrac{1-\cos\alpha}{1+\cos\alpha}$ gives

$$\cos\alpha=\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}=\frac{1-t^2}{1+t^2}$$

(3) $\sin\alpha$

[Sol] $\dfrac{\sin\alpha}{\cos\alpha}=\tan\alpha$

$$\sin\alpha=\tan\alpha\cos\alpha$$

$$=\frac{2t}{1-t^2}\cdot\frac{1-t^2}{1+t^2}$$

$$=\frac{2t}{1+t^2}$$

2. Using the results of the previous three exercises, evaluate each of the following expressions.

(1) Evaluate $\sin 2x + \cos 2x$ when $\tan x = -2$.

[Sol] Letting $\tan x = t$,

$$\sin 2x = \frac{2t}{1+t^2}, \cos 2x = \frac{1-t^2}{1+t^2}$$

$$\therefore \sin 2x + \cos 2x = \frac{2t + 1 - t^2}{1+t^2}$$

$$= \frac{-4+1-4}{1+4} = -\frac{7}{5}$$

Note: We are merely doubling the angle, from x to $2x$, similar to side a, when we doubled $\frac{\alpha}{2}$ to α.

(2) Evaluate $(\sin x + \cos x)^2$ when $\tan x = \frac{1}{3}$.

[Sol] $(\sin x + \cos x)^2 = \sin^2 x + 2\sin x \cos x + \cos^2 x$

$$= 1 + \sin 2x$$

$$= 1 + \frac{\frac{2}{3}}{1 + \frac{1}{9}}$$

$$= \frac{8}{5}$$

M 120 a

Addition Theorem 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. Perform the following calculations.

(1) $\sin 75^\circ$

[Sol] $\sin 75^\circ = \sin(45^\circ + 30^\circ)$

$$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

(2) $\cos 105^\circ$

[Sol] $\cos 105^\circ = \cos(60^\circ + 45^\circ)$

$$= \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ$$

$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

2. Obtain the values of $\sin 2\alpha$, $\cos 2\alpha$ and $\tan 2\alpha$ when $\cos\alpha = -\frac{2}{\sqrt{5}}$.
$\left(\pi < \alpha < \frac{3}{2}\pi\right)$

[Sol] Since $\pi < \alpha < \frac{3}{2}\pi$, $\sin\alpha = -\frac{1}{\sqrt{5}}$ and $\tan\alpha = \frac{1}{2}$.

$$\therefore \sin 2\alpha = 2\sin\alpha \cos\alpha = 2 \cdot \left(-\frac{1}{\sqrt{5}}\right) \cdot \left(-\frac{2}{\sqrt{5}}\right) = \frac{4}{5}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \left(\frac{4}{5}\right) - \left(\frac{1}{5}\right) = \frac{3}{5}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha} = \frac{2 \cdot \left(\frac{1}{2}\right)}{1 - \left(\frac{1}{2}\right)^2} = \frac{4}{3}$$

3. Evaluate the following expressions.

(1) $\sin 22.5°$

[Sol] Using the half-angle formulas,

$$\sin^2 22.5° = \frac{1-\cos 45°}{2} = \frac{1-\frac{\sqrt{2}}{2}}{2} = \frac{2-\sqrt{2}}{4}$$

Since $\sin 22.5° > 0$, $\sin 22.5° = \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{2}$

(2) $\tan 22.5°$

[Sol] $\tan^2 22.5° = \frac{1-\cos 45°}{1+\cos 45°} = \frac{2-\sqrt{2}}{2+\sqrt{2}} = 3-2\sqrt{2}$

Since $\tan 22.5° > 0$,

$\tan 22.5° = \sqrt{3-2\sqrt{2}} = \sqrt{2}-1$

4. Evaluate $\sin\frac{\alpha}{2}$ and $\cos\frac{\alpha}{2}$ when $\tan\alpha = -\frac{\sqrt{7}}{3}$. $\left(\frac{3}{2}\pi < \alpha < 2\pi\right)$

[Sol] Since $\frac{3}{2}\pi < \alpha < 2\pi$, $\cos\alpha = \frac{3}{4}$.

Since $\frac{3}{4}\pi < \frac{\alpha}{2} < \pi$, $\sin\frac{\alpha}{2} > 0$ and $\cos\frac{\alpha}{2} < 0$.

$$\therefore\ \sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-\frac{3}{4}}{2}} = \frac{\sqrt{2}}{4}$$

$$\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}} = -\sqrt{\frac{1+\frac{3}{4}}{2}} = -\frac{\sqrt{14}}{4}$$

Addition Theorem 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

1. Fill in the following boxes.

The Addition Theorem states these four formulas:

$$\begin{cases} \sin(\alpha+\beta) = \sin\alpha\cos\beta + \boxed{\cos\alpha\sin\beta} & \cdots\cdots ① \\ \sin(\alpha-\beta) = \sin\alpha\cos\beta - \boxed{\cos\alpha\sin\beta} & \cdots\cdots ② \\ \cos(\alpha+\beta) = \cos\alpha\cos\beta - \boxed{\sin\alpha\sin\beta} & \cdots\cdots ③ \\ \cos(\alpha-\beta) = \cos\alpha\cos\beta + \boxed{\sin\alpha\sin\beta} & \cdots\cdots ④ \end{cases}$$

① + ② gives

$$\sin(\alpha+\beta)+\sin(\alpha-\beta) = \boxed{2\sin\alpha\cos\beta}$$

① − ② gives

$$\sin(\alpha+\beta)-\sin(\alpha-\beta) = \boxed{2\cos\alpha\sin\beta}$$

③ + ④ gives

$$\cos(\alpha+\beta)+\cos(\alpha-\beta) = \boxed{2\cos\alpha\cos\beta}$$

③ − ④ gives

$$\cos(\alpha+\beta)-\cos(\alpha-\beta) = \boxed{-2\sin\alpha\sin\beta}$$

Product-to-Sum Formulas

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$

$$\sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha+\beta)-\cos(\alpha-\beta)]$$

Note

The Product-to-Sum Formulas express the product of two trigonometric values as a sum (or difference).

M 121 b

2. Convert each of the following products into a sum or difference using the Product-to-Sum Formulas.

Ex.

$$\sin 4\theta \cos 2\theta = \frac{1}{2}\sin(4\theta + 2\theta) + \frac{1}{2}\sin(4\theta - 2\theta)$$
$$= \frac{1}{2}\sin 6\theta + \frac{1}{2}\sin 2\theta$$

(1) $\sin 7\theta \cos\theta = \frac{1}{2}\sin 8\theta + \frac{1}{2}\sin 6\theta$

(2) $\cos 5\theta \cos 3\theta = \frac{1}{2}\cos 8\theta + \frac{1}{2}\cos 2\theta$

(3) $\cos 8\theta \sin 4\theta = \frac{1}{2}\sin 12\theta - \frac{1}{2}\sin 4\theta$

(4) $\sin 6\theta \sin 2\theta = -\frac{1}{2}\cos 8\theta + \frac{1}{2}\cos 4\theta$

(5) $4\sin 35^\circ \cos 15^\circ = 2\boxed{\sin 50^\circ} + \boxed{2\sin 20^\circ}$

(6) $\cos 55^\circ \sin 12^\circ = \frac{1}{2}\sin 67^\circ - \frac{1}{2}\sin 43^\circ$

(7) $3\cos 55^\circ \cos 20^\circ = \frac{3}{2}\cos 75^\circ + \frac{3}{2}\cos 35^\circ$

(8) $-2\sin 44^\circ \sin 40^\circ = \cos 84^\circ - \cos 4^\circ$

Addition Theorem 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Evaluate each of the following expressions.

Ex. $\sin 75^\circ \cos 15^\circ + \cos 105^\circ \sin 15^\circ$

$= \frac{1}{2}[\sin 90^\circ + \sin 60^\circ] + \frac{1}{2}[\sin 120^\circ - \sin 90^\circ]$

$= \frac{1}{2}\sin 60^\circ + \frac{1}{2}\sin 120^\circ$

$= \frac{1}{2}\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

Complete and check your answers.

$\sin 35^\circ \cos 25^\circ + \cos 50^\circ \sin 40^\circ$

$= \frac{1}{2}[\boxed{\sin 60^\circ} + \boxed{\sin 10^\circ}] + \frac{1}{2}[\boxed{\sin 90^\circ} - \boxed{\sin 10^\circ}]$

$= \frac{1}{2}\sin 60^\circ + \frac{1}{2}\sin 90^\circ$

$= \frac{1}{2}\cdot\boxed{\frac{\sqrt{3}}{2}} + \frac{1}{2}\cdot\boxed{1} = \boxed{\frac{\sqrt{3}+2}{4}}$

Answers: $\sin 60^\circ$, $\sin 10^\circ$, $\sin 90^\circ$, $\sin 10^\circ$, $\frac{\sqrt{3}}{2}$, 1, $\frac{\sqrt{3}+2}{4}$

(1) $\cos 40^\circ \cos 20^\circ - \sin 100^\circ \sin 80^\circ$

$= \frac{1}{2}[\cos 60^\circ + \cos 20^\circ] + \frac{1}{2}[\cos 180^\circ - \cos 20^\circ]$

$= \frac{1}{2}(\cos 60^\circ + \cos 180^\circ) = -\frac{1}{4}$

M 122 b

(2) $\cos 65° \sin 25° + \sin 35° \cos 5°$

$$= \frac{1}{2}[\sin 90° - \sin 40°] + \frac{1}{2}[\sin 40° + \sin 30°]$$

$$= \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{1}{2} = \frac{3}{4}$$

(3) $2\sin 105° \sin 15° = 2\cdot -\frac{1}{2}[\cos 120° - \cos 90°]$

$$= -\cos 120° + \cos 90° = \frac{1}{2}$$

(4) $-\sin 40° \cos 40° + \cos 70° \sin 10°$

$$= -\frac{1}{2}[\sin 80° + \sin 0°] + \frac{1}{2}[\sin 80° - \sin 60°]$$

$$= -\frac{\sqrt{3}}{4}$$

(5) $-\cos 40° \cos 20° + \sin 25° \sin 5°$

$$= -\frac{1}{2}[\cos 60° + \cos 20°] - \frac{1}{2}[\cos 30° - \cos 20°]$$

$$= -\frac{1}{2}\cdot\frac{1}{2} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2} = -\frac{\sqrt{3}+1}{4}$$

(6) $-6\sin 25° \cos 25° - 6\cos 115° \sin 65°$

$$= -6\cdot\frac{1}{2}[\sin 50° + \sin 0°] - 6\cdot\frac{1}{2}[\sin 180° - \sin 50°]$$

$$= 0$$

Addition Theorem 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

Convert each of the following products into a sum or difference of positive angles, using the Product-to-Sum Formulas and trigonometric properties. ($\theta > 0$)

Ex. $\cos 2\theta \sin 5\theta = \frac{1}{2}\sin 7\theta - \frac{1}{2}\sin(-3\theta)$

$= \frac{1}{2}\sin 7\theta + \frac{1}{2}\sin 3\theta$

$\sin(-\theta) = -\sin\theta$

(1) $2\sin 2\theta \cos 3\theta = \sin 5\theta + \sin(-\theta)$

$= \sin 5\theta - \sin\theta$

(2) $2\sin\theta \sin 3\theta = -\cos 4\theta + \cos(-2\theta)$

$= -\cos 4\theta + \cos 2\theta$

(3) $\sin 3\theta \sin 5\theta = -\frac{1}{2}\cos 8\theta + \frac{1}{2}\cos(-2\theta)$

$= -\frac{1}{2}\cos 8\theta + \frac{1}{2}\cos 2\theta$

(4) $\cos 8^\circ \cos 24^\circ = \frac{1}{2}[\cos 32^\circ + \cos(-16^\circ)]$

$= \frac{1}{2}\cos 32^\circ + \frac{1}{2}\cos(-16^\circ) = \frac{1}{2}\cos 32^\circ + \frac{1}{2}\cos 16^\circ$

M 123 b

(5) $3\cos 39^\circ \sin 49^\circ$

$$= 3 \cdot \frac{1}{2}[\sin 88^\circ - \sin(-10^\circ)]$$

$$= \frac{3}{2}\sin 88^\circ - \frac{3}{2}\sin(-10^\circ) = \frac{3}{2}\sin 88^\circ + \frac{3}{2}\sin 10^\circ$$

(6) $\sin 64^\circ \sin 16^\circ + \cos 18^\circ \sin 52^\circ$

$$= -\frac{1}{2}[\cos 80^\circ - \cos 48^\circ] + \frac{1}{2}[\sin 70^\circ - \sin(-34^\circ)]$$

$$= -\frac{1}{2}\cos 80^\circ + \frac{1}{2}\cos 48^\circ + \frac{1}{2}\sin 70^\circ + \frac{1}{2}\sin 34^\circ$$

(7) $\cos 54^\circ \cos 31^\circ + \sin 25^\circ \sin 40^\circ$

$$= \frac{1}{2}[\cos 85^\circ + \cos 23^\circ] - \frac{1}{2}[\cos 65^\circ - \cos(-15^\circ)]$$

$$= \frac{1}{2}\cos 85^\circ + \frac{1}{2}\cos 23^\circ - \frac{1}{2}\cos 65^\circ + \frac{1}{2}\cos 15^\circ$$

(8) $\sin 10^\circ \cos 20^\circ - \cos 35^\circ \sin 55^\circ$

$$= \frac{1}{2}[\sin 30^\circ + \sin(-10^\circ)] - \frac{1}{2}[\sin 90^\circ - \sin(-20^\circ)]$$

$$= \frac{1}{2}\sin 30^\circ + \frac{1}{2}\sin(-10^\circ) - \frac{1}{2}\sin 90^\circ + \frac{1}{2}\sin(-20^\circ)$$

$$= \frac{1}{4} - \frac{1}{2}\sin 10^\circ - \frac{1}{2} - \frac{1}{2}\sin 20^\circ$$

$$= -\frac{1}{4} - \frac{1}{2}\sin 10^\circ - \frac{1}{2}\sin 20^\circ$$

Addition Theorem 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

1. Fill in the following boxes.

From the Addition Theorem Formulas we have proven on page M121:

$$\begin{cases} \sin(\alpha+\beta) + \sin(\alpha-\beta) = \boxed{2\sin\alpha\cos\beta} & \cdots\cdots ① \\ \sin(\alpha+\beta) - \sin(\alpha-\beta) = \boxed{2\cos\alpha\sin\beta} & \cdots\cdots ② \\ \cos(\alpha+\beta) + \cos(\alpha-\beta) = \boxed{2\cos\alpha\cos\beta} & \cdots\cdots ③ \\ \cos(\alpha+\beta) - \cos(\alpha-\beta) = \boxed{-2\sin\alpha\sin\beta} & \cdots\cdots ④ \end{cases}$$

Letting $(\alpha+\beta) = A$ and $(\alpha-\beta) = B$,

$$\alpha = \frac{A+B}{2},\ \beta = \boxed{\frac{A-B}{2}}$$

Substituting these values into ①–④,

From ①, $\sin A + \sin B = \boxed{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}$

From ②, $\sin A - \sin B = \boxed{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}$

From ③, $\cos A + \cos B = \boxed{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}$

From ④, $\cos A - \cos B = \boxed{-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}}$

Sum-to-Product Formulas

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$
$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

Note

The Sum-to-Product Formulas express the sum (or difference) of two trigonometric values as a product.

2. Convert each of the following sums or differences into a product using the Sum-to-Product Formulas.

Ex. $$\boxed{\begin{aligned}\sin 5\theta + \sin 3\theta &= 2\sin\frac{5\theta+3\theta}{2}\cos\frac{5\theta-3\theta}{2}\\ &= 2\sin 4\theta\cos\theta\end{aligned}}$$

(1) $\sin 7\theta + \sin 3\theta = 2\sin\frac{\boxed{7\theta+3\theta}}{2}\cos\frac{\boxed{7\theta-3\theta}}{2}$

$= \boxed{2\sin 5\theta\cos 2\theta}$

(2) $\cos 3\theta + \cos\theta = 2\cos 2\theta\cos\theta$

(3) $\cos 7\theta - \cos 5\theta = -2\sin 6\theta\sin\theta$

(4) $\sin 78° - \sin 10° = 2\cos 44°\sin 34°$

(5) $\cos 73° - \cos 5° = -2\sin 39°\sin 34°$

(6) $3\sin 28° + 3\sin 12° = 3(\sin 28° + \sin 12°) = 6\sin 20°\cos 8°$

(7) $\frac{1}{2}\cos 54° + \frac{1}{2}\cos 22° = \frac{1}{2}(\cos 54° + \cos 22°)$

$= \cos 38°\cos 16°$

Addition Theorem 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Convert each of the following sums or differences into a product, using the Sum-to-Product Formulas and trigonometric properties.

Ex. $$\sin x + \sin(x-2y) = 2\sin\frac{x+(x-2y)}{2}\cos\frac{x-(x-2y)}{2}$$
$$= 2\sin(x-y)\cos y$$

(1) $$\cos(6x-2y) + \cos y = 2\cos\frac{(6x-2y)+y}{2}\cos\frac{(6x-2y)-y}{2}$$
$$= 2\cos\left(3x-\frac{1}{2}y\right)\cos\left(3x-\frac{3}{2}y\right)$$

(2) $$\cos(3x+5y) - \cos(x-y) = -2\sin\frac{(3x+5y)+(x-y)}{2}\sin\frac{(3x+5y)-(x-y)}{2}$$
$$= -2\sin\frac{4x+4y}{2}\sin\frac{2x+6y}{2}$$
$$= -2\sin(2x+2y)\sin(x+3y)$$

(3) $$\sin x + \cos x = \sin x + \sin\left(\frac{\pi}{2}-x\right)$$
$$= 2\sin\frac{x+\left(\boxed{\frac{\pi}{2}}-x\right)}{2}\cos\frac{x-\left(\boxed{\frac{\pi}{2}}-x\right)}{2}$$
$$= 2\sin\boxed{\frac{\pi}{4}}\cos\left(x-\boxed{\frac{\pi}{4}}\right) = \sqrt{2}\cos\left(x-\boxed{\frac{\pi}{4}}\right)$$

M 125 b

(4) $\sin x + \cos x = \cos\left(\boxed{\frac{\pi}{2}} - x\right) + \cos x$

$$= 2\cos\frac{\left(\boxed{\frac{\pi}{2}} - x\right) + x}{2}\cos\frac{\left(\boxed{\frac{\pi}{2}} - x\right) - x}{2}$$

$$= 2\cos\boxed{\frac{\pi}{4}}\cos\left(\boxed{\frac{\pi}{4}} - x\right) = \sqrt{2}\cos\left(\boxed{\frac{\pi}{4}} - x\right)$$

(5) $\sin 3x + \cos x = \sin 3x + \sin\left(\frac{\pi}{2} - x\right)$

$$= 2\sin\frac{3x + \left(\frac{\pi}{2} - x\right)}{2}\cos\frac{3x - \left(\frac{\pi}{2} - x\right)}{2}$$

$$= 2\sin\left(x + \frac{\pi}{4}\right)\cos\left(2x - \frac{\pi}{4}\right)$$

(6) $\cos 7x - \sin 3x = \cos 7x - \cos\left(\frac{\pi}{2} - 3x\right)$

$$= -2\sin\frac{7x + \left(\frac{\pi}{2} - 3x\right)}{2}\sin\frac{7x - \left(\frac{\pi}{2} - 3x\right)}{2}$$

$$= -2\sin\left(2x + \frac{\pi}{4}\right)\sin\left(5x - \frac{\pi}{4}\right)$$

Addition Theorem 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. Convert the following sum into a product.

$\cos x + \cos 3x + \cos 5x + \cos 7x$

$= (\cos x + \cos 3x) + (\cos 5x + \cos 7x)$

$= (2\cos 2x \cdot \cos x) + (2\cos 6x \cdot \cos x)$

$= 2\cos x(\cos 2x + \cos 6x)$

$= 2\cos x\left(2\cos\frac{2x+6x}{2}\cos\frac{2x-6x}{2}\right)$

$= 2\cos x[2\cos 4x \cdot \cos(-2x)] = 4\cos x(\cos 4x \cdot \cos 2x)$

2. Evaluate the following expressions.

(1) $\cos 80° + 2\cos 20° \cos 80°$

$= \cos 80° + 2\cdot\frac{1}{2}[\cos 100° + \cos(-60°)] = \cos 80° + \cos 100° + \cos 60°$

$= \left(2\cos\frac{80°+100°}{2}\cos\frac{80°-100°}{2}\right) + \frac{1}{2} = 2\cos 90° \cos 10° + \frac{1}{2}$

$= 0 + \frac{1}{2} = \frac{1}{2}$

(2) $\sin 10° \cos 10° - \cos^2 35°$

$= \frac{1}{2}[\sin 20° + \sin 0°] - \frac{1}{2}[\cos 70° + \cos 0°]$

$= \frac{1}{2}\sin 20° + \frac{1}{2}\cdot 0 - \frac{1}{2}\cos 70° - \frac{1}{2}\cdot 1$

$= \frac{1}{2}\sin 20° + 0 - \frac{1}{2}\sin 20° - \frac{1}{2}$

$= -\frac{1}{2}$

3. Convert the following product into a sum or difference.

$$4\cos 2\theta \cos 4\theta \cos 6\theta = 2(\cos 2\theta \cdot \cos 4\theta) \cdot 2\cos 6\theta$$

$$= 2 \cdot \frac{1}{2}[\cos 6\theta + \cos 2\theta] \cdot 2\cos 6\theta$$

$$= (\cos 6\theta + \cos 2\theta) \cdot 2\cos 6\theta$$

$$= 2\cos^2 6\theta + 2\cos 2\theta \cos 6\theta$$

$$= \cos 12\theta + \cos 0 + \cos 8\theta + \cos 4\theta$$

$$= 1 + \cos 12\theta + \cos 8\theta + \cos 4\theta$$

4. Evaluate the following expressions.

(1) $\sin 15° \sin 75° \cos 22.5° \sin 22.5°$

$$= (\sin 15° \sin 75°)(\cos 22.5° \sin 22.5°)$$

$$= \left\{-\frac{1}{2}[\cos 90° - \cos(-60°)]\right\} \cdot \left\{\frac{1}{2}[\sin 45° - \sin(0°)]\right\}$$

$$= \frac{1}{2}\cos 60° \cdot \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{16}$$

(2) $\cos 65° \cos 175° - \cos 185° \cos 65°$

$$= \frac{1}{2}[\cos 240° + \cos(-110°)] - \frac{1}{2}[\cos 250° + \cos 120°]$$

$$= \frac{1}{2}\cos 240° + \frac{1}{2}\cos 110° - \frac{1}{2}\cos 250° - \frac{1}{2}\cos 120°$$

$$= \frac{1}{2}\cos 120° + \frac{1}{2}\cos 110° - \frac{1}{2}\cos 110° - \frac{1}{2}\cos 120°$$

$$= 0$$

M 127 a Addition Theorem 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. Simplify each of the following expressions.

(1) $\sin\left(\frac{\pi}{6}+x\right)+\sin\left(\frac{\pi}{6}-x\right)\cancel{-\cos x}$

$=2\sin\frac{\pi}{6}\cos x-\cos x$

$=\cos x-\cos x$

$=0$

(2) $\sin x+\sin\left(x+\frac{2}{3}\pi\right)+\sin\left(x+\frac{4}{3}\pi\right)$

$=\sin x+2\sin(x+\pi)\cos\frac{\pi}{3}$

$=\sin x-\sin x$

$=0$

(3) $4\cos x\cos\left(x+\frac{2}{3}\pi\right)\cos\left(x+\frac{4}{3}\pi\right)$

$=2\cos x\left[\cos(2x+2\pi)+\cos\frac{2}{3}\pi\right]$

$=2\cos x\left(\cos 2x-\frac{1}{2}\right)$

$=2\cos x\cos 2x-\cos x$

$=\cos 3x+\cos x-\cos x$

$=\cos 3x$

M 127 b

2. Prove each of the following equalities.

(1) $\cos(\alpha+\beta)\cos(\alpha-\beta)=\cos^2\alpha-\sin^2\beta$

[Proof] $\cos(\alpha+\beta)\cos(\alpha-\beta)=\frac{1}{2}\cos 2\alpha+\frac{1}{2}\boxed{\cos 2\beta}$

$$=\cos^2\alpha-\frac{1}{2}+\frac{1}{2}-\sin^2\beta$$

$$=\cos^2\alpha-\sin^2\beta$$

(2) $\sin(\alpha+\beta)\cos(\alpha-\beta)=\sin\alpha\cos\alpha+\sin\beta\cos\beta$

[Proof] $\sin(\alpha+\beta)\cos(\alpha-\beta)=\frac{1}{2}\sin 2\alpha+\frac{1}{2}\sin 2\beta$

$$=\frac{1}{2}(2\sin\alpha\cos\alpha)+\frac{1}{2}(2\sin\beta\cos\beta)$$

$$=\sin\alpha\cos\alpha+\sin\beta\cos\beta$$

(3) $\dfrac{\sin x+\sin y}{\cos x-\cos y}=-\dfrac{1}{\tan\dfrac{x-y}{2}}$

[Proof] $\dfrac{\sin x+\sin y}{\cos x-\cos y}=\dfrac{2\sin\dfrac{x+y}{2}\cos\dfrac{x-y}{2}}{-2\sin\dfrac{x+y}{2}\sin\dfrac{x-y}{2}}=-\dfrac{\cos\dfrac{x-y}{2}}{\sin\dfrac{x-y}{2}}$

$$=-\frac{1}{\tan\dfrac{x-y}{2}}$$

M 128 a

Addition Theorem 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Using the trigonometric properties and the formulas already learned, solve each of the following equations. $(0 \leq x < 2\pi)$

(1) $\sin x = \sin 2x$

[Sol] $\sin x - \sin 2x = 0$

$\sin x - 2\sin x \cos x = 0$

$\sin x(1 - 2\cos x) = 0$

$\therefore \sin x = 0, \ \cos x = \frac{1}{2}$

$\therefore x = 0, \pi, \frac{\pi}{3}, \frac{5}{3}\pi$

(2) $\sin 2x = \cos x$

[Sol] $\sin 2x - \cos x = 0$

$2\sin x \cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\therefore \cos x = 0, \ \sin x = \frac{1}{2}$

$\therefore x = \frac{\pi}{2}, \frac{3}{2}\pi, \frac{\pi}{6}, \frac{5}{6}\pi$

(3) $3\sin x - 1 = \cos 2x$

[Sol] $3\sin x - 1 - \cos 2x = 0$

$3\sin x - 1 - (1 - 2\sin^2 x) = 0$

$2\sin^2 x + 3\sin x - 2 = 0$

$(2\sin x - 1)(\sin x + 2) = 0$

$\sin x = \frac{1}{2}$, since $\sin x \neq -2$

$\therefore x = \frac{\pi}{6}, \frac{5}{6}\pi$

(4) $\cos 2x - \cos x = 0$

[Sol] $(2\cos^2 x - 1) - \cos x = 0$

$2\cos^2 x - \cos x - 1 = 0$

$(2\cos x + 1)(\cos x - 1) = 0$

$\therefore \cos x = -\frac{1}{2}, 1$

$\therefore x = 0, \frac{2}{3}\pi, \frac{4}{3}\pi$

(5) $\cos 2x - 2\sin x = 1$

[Sol] $(1 - 2\sin^2 x) - 2\sin x = 1$

$2\sin^2 x + 2\sin x = 0$

$2\sin x(\sin x + 1) = 0$

$\therefore \sin x = 0, -1$

$\therefore x = 0, \pi, \frac{3}{2}\pi$

(6) $\cos 2x - 3\cos x + 2 = 0$

[Sol] $(2\cos^2 x - 1) - 3\cos x + 2 = 0$

$2\cos^2 x - 3\cos x + 1 = 0$

$(2\cos x - 1)(\cos x - 1) = 0$

$\therefore \cos x = \frac{1}{2}, 1$

$\therefore x = \frac{\pi}{3}, \frac{5}{3}\pi, 0$

M 129 a

Addition Theorem 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Solve each of the following equations.

(1) $\sin x + \sin 2x + \sin 3x = 0 \qquad (0 \leq x < \pi)$

[Sol] $(\sin x + \sin 3x) + \sin 2x = 0$

$$\left(2\sin\frac{x+\boxed{3x}}{2}\cos\frac{x-\boxed{3x}}{2}\right) + \sin 2x = 0$$

$$(2\sin 2x\boxed{\cos x}) + \sin 2x = 0$$

$$\sin 2x(\boxed{2\cos x + 1}) = 0$$

$$\therefore \sin 2x = 0,\ \cos x = \boxed{-\frac{1}{2}}$$

Since $0 \leq x < \pi,\ 0 \leq 2x < \boxed{2\pi}$

$$\therefore x = \boxed{0}, \boxed{\frac{\pi}{2}}, \boxed{\frac{2}{3}\pi}$$

(2) $\cos x + \cos 2x + \cos 3x = 0 \qquad (0 \leq x < \pi)$

[Sol] $2\cos 2x\cos x + \cos 2x = 0$

$$\cos 2x(2\cos x + 1) = 0$$

$$\therefore \cos 2x = 0,\ \cos x = -\frac{1}{2}$$

$$\therefore x = \frac{\pi}{4}, \frac{3}{4}\pi, \frac{2}{3}\pi$$

(3) $\cos 7x + \cos 3x = \cos 2x \qquad \left(0 \leq x \leq \frac{\pi}{2}\right)$

[Sol] $(\cos 7x + \cos 3x) - \cos 2x = 0$

$$2\cos 5x\cos 2x - \cos 2x = 0$$

$$\cos 2x(2\cos 5x - 1) = 0$$

$$\therefore \cos 2x = 0,\ \cos 5x = \frac{1}{2}$$

Since $0 \leq x \leq \frac{\pi}{2},\ 0 \leq 2x \leq \pi$ and $0 \leq 5x \leq \frac{5}{2}\pi$

$$\therefore x = \frac{\pi}{4}, \frac{\pi}{15}, \frac{\pi}{3}, \frac{7}{15}\pi$$

M 129 b

(4) $\cos x \cos 2x = \cos 3x \cos 4x \qquad \left(0 \leq x < \dfrac{\pi}{2}\right)$

[Sol] $\dfrac{1}{2}[\cos 3x + \cos(-x)] = \dfrac{1}{2}[\cos 7x + \cos(-x)]$

$\cos 3x + \cos x = \cos 7x + \cos x$

$\cos 3x - \cos 7x = 0$

$-2\sin\dfrac{3x+7x}{2}\ \sin\dfrac{3x-7x}{2} = 0$

$-2\sin 5x \sin(-2x) = 0$

$2\sin 5x \sin 2x = 0$

$\therefore\ \sin 5x = 0,\ \sin 2x = 0$

Since $0 \leq x < \dfrac{\pi}{2}$, $0 \leq 5x < \dfrac{5}{2}\pi$ and $0 \leq 2x < \pi$

$\sin 5x = 0 \qquad 5x = 0, \pi, 2\pi$

$\sin 2x = 0 \qquad 2x = 0$

$\therefore\ x = 0, \dfrac{\pi}{5}, \dfrac{2}{5}\pi$

(5) $\cos x \cos 3x = \sin 5x \sin 7x \qquad \left(0 \leq x < \dfrac{\pi}{4}\right)$

[Sol] $\dfrac{1}{2}[\cos 4x + \cos(-2x)] = -\dfrac{1}{2}[\cos 12x - \cos(-2x)]$

$\dfrac{1}{2}\cos 4x + \dfrac{1}{2}\cos 2x = -\dfrac{1}{2}\cos 12x + \dfrac{1}{2}\cos 2x$

$\cos 4x = -\cos 12x$

$\cos 4x + \cos 12x = 0$

$2\cos\dfrac{4x+12x}{2}\ \cos\dfrac{4x-12x}{2} = 0$

$2\cos 8x \cos 4x = 0$

$\therefore\ \cos 8x = 0,\ \cos 4x = 0$

Since $0 \leq x < \dfrac{\pi}{4}$, $0 \leq 8x < 2\pi$ and $0 \leq 4x < \pi$

$\cos 8x = 0 \qquad 8x = \dfrac{\pi}{2}, \dfrac{3}{2}\pi$

$\cos 4x = 0 \qquad 4x = \dfrac{\pi}{2}$

$\therefore\ x = \dfrac{\pi}{16}, \dfrac{3}{16}\pi, \dfrac{\pi}{8}$

Addition Theorem 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

1. Convert each of the following products into a sum or difference.

(1) $\cos 7\theta \cos 2\theta = \frac{1}{2}\cos 9\theta + \frac{1}{2}\cos 5\theta$

(2) $-2\sin 65^\circ \cos 18^\circ = 2\cdot -\frac{1}{2}[\sin 83^\circ + \sin 47^\circ]$

$= -\sin 83^\circ - \sin 47^\circ$

(3) $4\cos 40^\circ \cos 50^\circ - 4\sin 27.5^\circ \sin 17.5^\circ$

$= 4\cdot\frac{1}{2}[\cos 90^\circ + \cos(-10^\circ)] - 4\cdot -\frac{1}{2}[\cos 45^\circ - \cos 10^\circ]$

$= 2\cos 90^\circ + 2\cos 10^\circ + 2\cos 45^\circ - 2\cos 10^\circ$

$= \sqrt{2}$

2. Convert each of the following differences into a product.

(1) $\cos 4\theta - \cos 3\theta = -2\sin\frac{4\theta + 3\theta}{2}\sin\frac{4\theta - 3\theta}{2}$

$= -2\sin\frac{7}{2}\theta \sin\frac{1}{2}\theta$

(2) $-3\sin 35^\circ - 3\sin 15^\circ = -3\cdot(\sin 35^\circ + \sin 15^\circ)$

$= -3\cdot\left(2\sin\frac{35^\circ + 15^\circ}{2}\cos\frac{35^\circ - 15^\circ}{2}\right)$

$= -6\sin 25^\circ \cos 10^\circ$

3. Solve each of the following equations.

(1) $\sin x + \cos\frac{x}{2} = 0 \qquad (0 \leq x < 2\pi)$

[Sol] $\sin\left(2 \cdot \frac{x}{2}\right) + \cos\frac{x}{2} = 0$

$$2\sin\frac{x}{2}\cos\frac{x}{2} + \cos\frac{x}{2} = 0$$

$$\cos\frac{x}{2}\left(2\sin\frac{x}{2} + 1\right) = 0$$

$$\therefore \cos\frac{x}{2} = 0, \quad \sin\frac{x}{2} = -\frac{1}{2}$$

Since $0 \leq x < 2\pi$, $\sin\frac{x}{2} \geq 0$. Hence, $\sin\frac{x}{2} \neq -\frac{1}{2}$

Using $\cos\frac{x}{2} = 0$, $\quad \frac{x}{2} = \frac{\pi}{2}$

$$\therefore x = \pi$$

(2) $\cos 2x = -3\sin x + 2 \qquad (-\pi \leq x \leq \pi)$

[Sol] $\cos 2x + 3\sin x - 2 = 0$

$$(1 - 2\sin^2 x) + 3\sin x - 2 = 0$$

$$2\sin^2 x - 3\sin x + 1 = 0$$

$$(2\sin x - 1)(\sin x - 1) = 0$$

$$\therefore \sin x = \frac{1}{2}, 1$$

$$\therefore x = \frac{1}{6}\pi, \frac{5}{6}\pi, \frac{\pi}{2}$$

M 131 a Addition Theorem 3

Time : to : Date Name

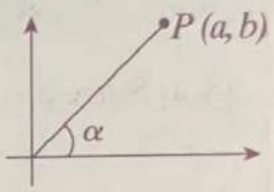
100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1~

So far, we have learned how to add (and subtract) trigonometric values of the same function, such as $\sin\alpha \pm \sin\beta$ or $\cos\alpha \pm \cos\beta$. We will now learn how to express $\sin\alpha + \cos\beta$.

1. Follow the formula and fill in the blanks.

Formula $a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}\sin(\theta+\alpha)$

(where θ is any angle, and α is any random angle that follows the properties on the right)



[Proof] $a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}\sin(\theta+\alpha)$

RHS $= \sqrt{a^2+b^2}\sin(\theta+\alpha) = \sqrt{a^2+b^2}\,(\boxed{\sin\theta\cos\alpha} + \cos\theta\sin\alpha)$

Thus, from the diagram, the following properties exist:

$$\sin\alpha = \boxed{\frac{b}{\sqrt{a^2+b^2}}} \quad \text{and} \quad \cos\alpha = \boxed{\frac{a}{\sqrt{a^2+b^2}}}$$

Therefore, $\sqrt{a^2+b^2}\sin(\theta+\alpha)$

$$= \sqrt{a^2+b^2}\left[(\sin\theta)\frac{a}{\sqrt{a^2+b^2}} + (\cos\theta)\frac{b}{\sqrt{a^2+b^2}}\right]$$

$$= a\sin\theta + b\cos\theta$$

$$\therefore \sqrt{a^2+b^2}\sin(\theta+\alpha) = a\sin\theta + b\cos\theta$$

Note

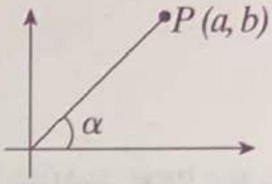
The LHS is the same as the RHS, but it is in terms of one trigonometric value instead of two.

M 131 b

Formula

$$a \sin\theta + b\cos\theta = \sqrt{a^2+b^2}\sin(\theta+\alpha)$$

(where α is the angle shown in the diagram on the right)



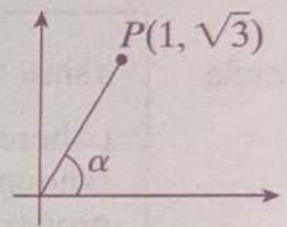
2. Convert each of the following expressions into the form $r\sin(\theta+\alpha)$. $(-\pi < \alpha < \pi)$

Ex. $\sin\theta + \sqrt{3}\cos\theta$

[Sol] Since $a = 1$ and $b = \sqrt{3}$, $\alpha = \frac{\pi}{3}$.

$$\therefore \sin\theta + \sqrt{3}\cos\theta = \sqrt{1+3}\sin\left(\theta+\frac{\pi}{3}\right)$$

$$= 2\sin\left(\theta+\frac{\pi}{3}\right)$$



(1) $\sin\theta + \cos\theta$

[Sol] Since $a = 1$ and $b = 1$, $\alpha = \frac{\pi}{4}$.

$$\therefore \sin\theta + \cos\theta = \sqrt{1+1}\sin\left(\theta+\frac{\pi}{4}\right)$$

$$= \sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)$$

(2) $\sin\theta - \cos\theta$

[Sol] Since $a = 1$ and $b = -1$, $\alpha = -\boxed{\frac{\pi}{4}}$.

$$\therefore \sin\theta - \cos\theta = \sqrt{1^2+(-1)^2}\sin\left(\theta-\frac{\pi}{4}\right)$$

$$= \sqrt{2}\sin\left(\theta-\frac{\pi}{4}\right)$$

KUMON

M 132 a　Addition Theorem 3

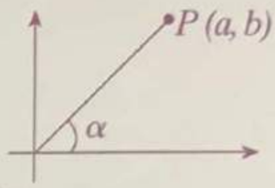
Time　:　to　:　　Date　　Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Formula

$a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}\sin(\theta+\alpha)$

(where θ is any angle, and α is any random angle that follows the properties on the right)



Convert each of the following expressions into the form $r\sin(kx+\alpha)$. $(-\pi<\alpha<\pi)$

Complete and check your answers.

$\sin 2x + \sqrt{3}\cos 2x$

[Sol] Since $a=1$ and $b=\sqrt{3}$, $\alpha = \boxed{\frac{\pi}{3}}$.

$\therefore\ \sin 2x + \sqrt{3}\cos 2x = 2\sin\left(2x + \boxed{\frac{\pi}{3}}\right)$

Answers: both are $\frac{\pi}{3}$

(1) $\sqrt{3}\sin 3x + \cos 3x$

[Sol] Since $a=\sqrt{3}$ and $b=1$, $\alpha=\frac{\pi}{6}$.

$\therefore\ \sqrt{3}\sin 3x + \cos 3x = 2\sin\left(3x+\frac{\pi}{6}\right)$

(2) $3\sin\frac{1}{2}x - \sqrt{3}\cos\frac{1}{2}x$

[Sol] Since $a=3$ and $b=-\sqrt{3}$, $\alpha=-\frac{\pi}{6}$.

$\therefore\ 3\sin\frac{1}{2}x - \sqrt{3}\cos\frac{1}{2}x = 2\sqrt{3}\sin\left(\frac{1}{2}x-\frac{\pi}{6}\right)$

(3) $\sqrt{3}\sin 2x + 2\cos^2 x - 1$

[Sol] $\sqrt{3}\sin 2x + (2\cos^2 x - 1) = \sqrt{3}\sin 2x + (\cos 2x)$

Since $a=\sqrt{3}$ and $b=1$, $\alpha=\frac{\pi}{6}$.

$\therefore\ \sqrt{3}\sin 2x + 2\cos^2 x - 1 = 2\sin\left(2x+\frac{\pi}{6}\right)$

M 132 b

(4) $2\sqrt{3}\sin\left(x+\frac{\pi}{6}\right)-4\sin x$

[Sol] $2\sqrt{3}\sin\left(x+\frac{\pi}{6}\right)-4\sin x = 2\sqrt{3}\left(\sin x \boxed{\cos\frac{\pi}{6}} + \cos x \boxed{\sin\frac{\pi}{6}}\right) - 4\sin x$

$$= 2\sqrt{3}\left(\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x\right) - 4\sin x$$

$$= -\sin x + \sqrt{3}\cos x$$

$$= \boxed{2\sin\left(x+\frac{2}{3}\pi\right)}$$

(5) $2\sin^2 x + \sin 2x - 1$

$= (1-\cos 2x) + \sin 2x - 1$

$= \sin 2x - \cos 2x$

Since $a = 1$ and $b = -1$, $\alpha = -\frac{\pi}{4}$.

$\therefore\ 2\sin^2 x + \sin 2x - 1 = \sqrt{2}\sin\left(2x - \frac{\pi}{4}\right)$

(6) $2\sin\left(3x+\frac{\pi}{6}\right)\cos\frac{\pi}{6} + \cos\left(3x+\frac{\pi}{6}\right)$

$= \sqrt{3}\sin\left(3x+\frac{\pi}{6}\right) + \cos\left(3x+\frac{\pi}{6}\right)$

Since $a = \sqrt{3}$ and $b = 1$, $\alpha = \frac{\pi}{6}$.

$\therefore\ 2\sin\left(3x+\frac{\pi}{6}\right)\cos\frac{\pi}{6} + \cos\left(3x+\frac{\pi}{6}\right) = 2\sin\left(3x+\frac{\pi}{3}\right)$

Addition Theorem 3

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Solve each of the following equations. $(0 \leq x < 2\pi)$

(1) $\sqrt{3}\sin x + \cos x = 0$

[Sol] Since $a = \sqrt{3}$ and $b = 1$, $\alpha = \frac{\pi}{6}$.

Thus, the new equation is: $2\sin\left(x + \frac{\pi}{6}\right) = 0$

$$\therefore \sin\left(x + \frac{\pi}{6}\right) = 0$$

From the condition of $0 \leq x < 2\pi$, $\frac{\pi}{6} \leq x + \boxed{\frac{\pi}{6}} < \boxed{\frac{13}{6}\pi}$

$x + \frac{\pi}{6} = \pi, 2\pi$

$$\therefore x = \boxed{\frac{5}{6}\pi}, \boxed{\frac{11}{6}\pi}$$

(2) $\sin x + \cos x = 1$

[Sol] Since $a = 1$ and $b = 1$, $\alpha = \frac{\pi}{4}$.

The new equation: $\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) = 1$

$$\therefore \sin\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

From the condition, $\frac{\pi}{4} \leq x + \frac{\pi}{4} < \frac{9}{4}\pi$

$x + \frac{\pi}{4} = \frac{\pi}{4}, \frac{3}{4}\pi$

$$\therefore x = 0, \frac{\pi}{2}$$

(3) $\sin x + \sqrt{3}\cos x = 1$

[Sol] Since $a = 1$ and $b = \sqrt{3}$, $\alpha = \frac{\pi}{3}$.

The new equation: $2\sin\left(x + \frac{\pi}{3}\right) = 1$

$$\therefore \sin\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$$

From the condition, $\frac{\pi}{3} \leq x + \frac{\pi}{3} < \frac{7}{3}\pi$

$x + \frac{\pi}{3} = \frac{5}{6}\pi, \frac{13}{6}\pi$

$$\therefore x = \frac{\pi}{2}, \frac{11}{6}\pi$$

(4) $\cos x - \sqrt{3}\sin x = \sqrt{3}$

[Sol] Since $a = -\sqrt{3}$ and $b = 1$, $\alpha = \frac{5}{6}\pi$.

The new equation: $2\sin\left(x + \frac{5}{6}\pi\right) = \sqrt{3}$

$\therefore\ \sin\left(x + \frac{5}{6}\pi\right) = \frac{\sqrt{3}}{2}$

From the condition, $\frac{5}{6}\pi \leq x + \frac{5}{6}\pi < \frac{17}{6}\pi$

$x + \frac{5}{6}\pi = \frac{7}{3}\pi, \frac{8}{3}\pi$

$\therefore\ x = \frac{3}{2}\pi, \frac{11}{6}\pi$

(5) $\sin\frac{x}{2} + \sqrt{3}\cos\frac{x}{2} = 1$

[Sol] Since $a = 1$ and $b = \sqrt{3}$, $\alpha = \frac{\pi}{3}$

The new equation: $2\sin\left(\frac{x}{2} + \frac{\pi}{3}\right) = 1$

$\therefore\ \sin\left(\frac{x}{2} + \frac{\pi}{3}\right) = \frac{1}{2}$

From the condition, $\frac{\pi}{3} \leq \frac{x}{2} + \frac{\pi}{3} < \frac{4}{3}\pi$

$\frac{x}{2} + \frac{\pi}{3} = \frac{5}{6}\pi$

$\therefore\ x = \pi$

(6) $\sin 2x - 2\sin^2 x + 1 = 0$

[Sol] $\sin 2x + (-2\sin^2 x + 1) = 0$

$\sin 2x + \cos 2x = 0$

Since $a = 1$ and $b = 1$, $\alpha = \frac{\pi}{4}$.

The new equation: $\sqrt{2}\sin\left(2x + \frac{\pi}{4}\right) = 0$

$\therefore\ \sin\left(2x + \frac{\pi}{4}\right) = 0$

From the condition, $\frac{\pi}{4} \leq 2x + \frac{\pi}{4} < \frac{17}{4}\pi$

$\therefore\ x = \frac{3}{8}\pi, \frac{7}{8}\pi, \frac{11}{8}\pi, \frac{15}{8}\pi$

Addition Theorem 3

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Solve each of the following inequalities. $(0 \leq x < 2\pi)$

(1) $\sin x < \cos x$

[Sol] $\sin x - \cos x < 0$,

$$\sin\left(x - \frac{\pi}{4}\right) < 0$$

From the condition, $-\frac{\pi}{4} \leq x - \frac{\pi}{4} < \frac{7}{4}\pi$

Solving the inequality in this range,

$-\frac{\pi}{4} \leq x - \frac{\pi}{4} < 0$, and $\pi < x - \frac{\pi}{4} < \frac{7}{4}\pi$

$\therefore\ 0 \leq x < \frac{\pi}{4}$ and $\frac{5}{4}\pi < x < 2\pi$

(2) $\sin x - \cos x > 1$

[Sol] $\sin\left(x - \frac{\pi}{4}\right) > \frac{1}{\sqrt{2}}$

From the condition, $-\frac{\pi}{4} \leq x - \frac{\pi}{4} < \frac{7}{4}\pi$

Solving the inequality in this range,

$\frac{\pi}{4} < x - \frac{\pi}{4} < \frac{3}{4}\pi$

$\therefore\ \frac{\pi}{2} < x < \pi$

M 134 b

(3) $\cos\frac{x}{2}+\sqrt{3}\sin\frac{x}{2}<0$

[Sol] $\sin\left(\frac{x}{2}+\frac{\pi}{6}\right)<0$

From the condition, $\frac{\pi}{6}\leq\frac{x}{2}+\frac{\pi}{6}<\frac{7}{6}\pi$

Solving the inequality in this range,

$\pi<\frac{x}{2}+\frac{\pi}{6}<\frac{7}{6}\pi$

$\therefore\ \frac{5}{3}\pi<x<2\pi$

(4) $\sqrt{3}\sin x\geq\sqrt{6}-3\cos x$

[Sol] $\sqrt{3}\sin x+3\cos x\geq\sqrt{6}$

$\sin\left(x+\frac{\pi}{3}\right)\geq\frac{\sqrt{2}}{2}$

From the condition, $\frac{\pi}{3}\leq x+\frac{\pi}{3}<\frac{7}{3}\pi$

Solving the inequality in this range,

$\frac{\pi}{3}\leq x+\frac{\pi}{3}\leq\frac{3}{4}\pi,\quad \frac{9}{4}\pi\leq x+\frac{\pi}{3}<\frac{7}{3}\pi$

$0\leq x\leq\frac{5}{12}\pi,\qquad \frac{23}{12}\pi\leq x<2\pi$

(5) $1-2\sin^2 x\leq\sin 2x-\sqrt{2}$

[Sol] $\cos 2x\leq\sin 2x-\sqrt{2}$

$\sin 2x-\cos 2x\geq\sqrt{2}$

$\sqrt{2}\sin\left(2x-\frac{\pi}{4}\right)\geq\sqrt{2}$

$\sin\left(2x-\frac{\pi}{4}\right)\geq 1$

$x=\frac{3\pi}{8},\frac{11\pi}{8}$

Addition Theorem 3

Time : to : Date Name

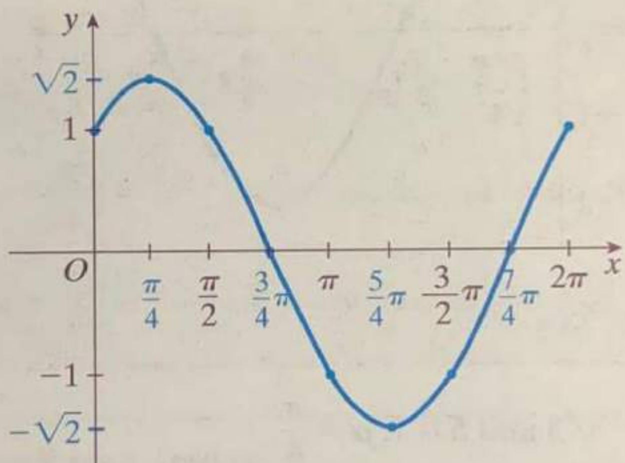
100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

In each of the following exercises, convert the given equation into the form $y = r\sin(kx + \alpha)$ and then graph the equation on the axes provided.

(1) $y = \sin x + \cos x$

[Sol] Since $a = 1$ and $b = 1$, $\alpha = \dfrac{\pi}{4}$.

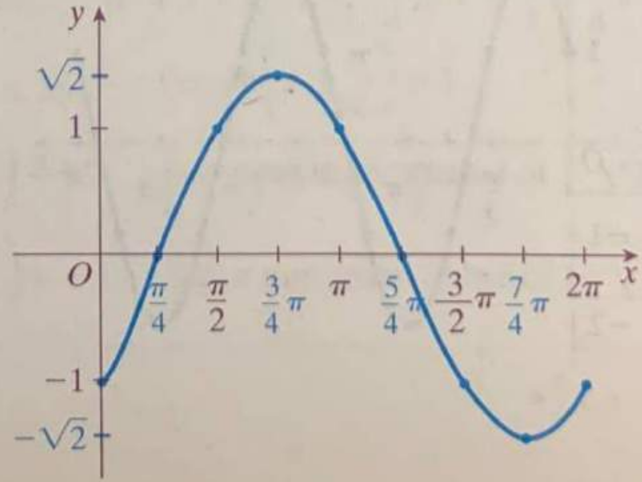
$$\therefore\ \sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$$



(2) $y = \sin x - \cos x$

[Sol] Since $a = 1$ and $b = -1$, $\alpha = -\dfrac{\pi}{4}$.

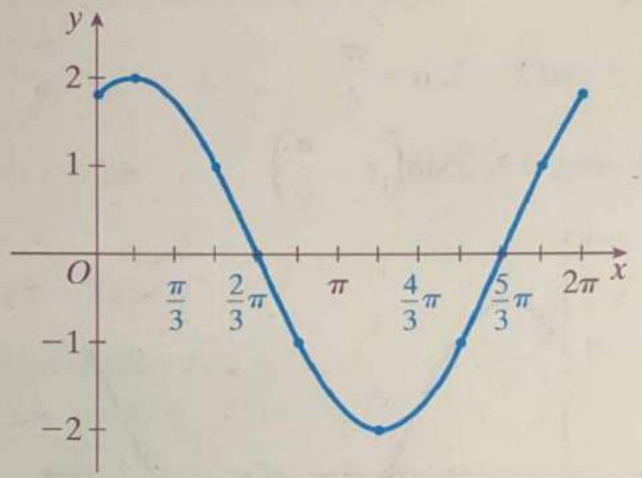
$$\therefore\ \sin x - \cos x = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$$



(3) $y = \sin x + \sqrt{3}\cos x$

[Sol] Since $a = 1$ and $b = \sqrt{3}$, $\alpha = \frac{\pi}{3}$.

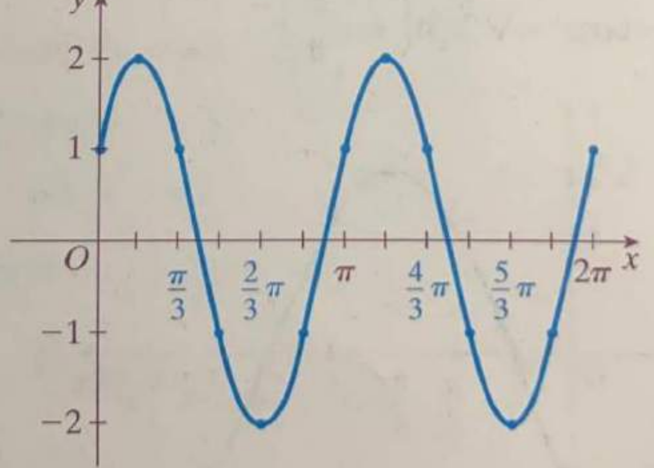
$$\therefore\ \sin x + \sqrt{3}\cos x = 2\sin\left(x + \frac{\pi}{3}\right)$$



(4) $y = \sqrt{3}\sin 2x + \cos 2x$

[Sol] Since $a = \sqrt{3}$ and $b = 1$, $\alpha = \frac{\pi}{6}$.

$$\therefore\ \sqrt{3}\sin 2x + \cos 2x = 2\sin\left(2x + \frac{\pi}{6}\right)$$



Addition Theorem 3

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1~

Given that $0 \leq x < 2\pi$, find the maximum and minimum values of each of the following functions, and state the value of x at which each occurs.

Ex. $y = \sin x - \cos x$

[Sol] Since $a = 1$ and $b = -1$, $\alpha = -\dfrac{\pi}{4}$.

$\therefore\ y = \sin x - \cos x = \sqrt{2}\sin\left(x - \dfrac{\pi}{4}\right)$

From the condition of $0 \leq x < 2\pi$, $-\dfrac{\pi}{4} \leq x - \dfrac{\pi}{4} < \dfrac{7}{4}\pi$

Since $0 \leq x < 2\pi$, $\quad -1 \leq \sin\left(x - \dfrac{\pi}{4}\right) \leq 1$

When $\sin\left(x - \dfrac{\pi}{4}\right) = 1$, y has a maximum value of $\sqrt{2}$, at $x = \dfrac{3}{4}\pi$.

When $\sin\left(x - \dfrac{\pi}{4}\right) = -1$, y has a minimum value of $-\sqrt{2}$, at $x = \dfrac{7}{4}\pi$.

Complete and check your answers.

$y = \sin x + \cos x$

[Sol] Since $a = 1$ and $b = 1$, $\alpha = \boxed{\dfrac{\pi}{4}}$.

$\therefore\ y = \sin x + \cos x = \sqrt{2}\sin\left(x + \boxed{\dfrac{\pi}{4}}\right)$

From the condition of $0 \leq x < 2\pi$, $\dfrac{\pi}{4} \leq x + \dfrac{\pi}{4} < \dfrac{9}{4}\pi$

Since $0 \leq x < 2\pi$, $\quad -1 \leq \sin\left(x + \dfrac{\pi}{4}\right) \leq 1$

When $\sin\left(x + \dfrac{\pi}{4}\right) = 1$, y has a maximum value of $\boxed{\sqrt{2}}$, at $x = \boxed{\dfrac{\pi}{4}}$.

When $\sin\left(x + \dfrac{\pi}{4}\right) = -1$, y has a minimum value of $\boxed{-\sqrt{2}}$, at $x = \boxed{\dfrac{5}{4}\pi}$.

Answers: $\frac{\pi}{4}, \frac{\pi}{4}, \sqrt{2}, \frac{\pi}{4}, -\sqrt{2}, \frac{5}{4}\pi$

M 136 b

(1) $y = \sin x + \sqrt{3}\cos x$

[Sol] Since $a = 1$ and $b = \sqrt{3}$, $\alpha = \frac{\pi}{3}$.

$$\therefore\ y = \sin x + \sqrt{3}\cos x = 2\sin\left(x + \frac{\pi}{3}\right)$$

From the condition, $\frac{\pi}{3} \le x + \frac{\pi}{3} < \frac{7}{3}\pi$

Since $0 \le x < 2\pi$, $-1 \le \sin\left(x + \frac{\pi}{3}\right) \le 1$

When $\sin\left(x + \frac{\pi}{3}\right) = 1$, y has a maximum value of 2, at $x = \frac{\pi}{6}$.

When $\sin\left(x + \frac{\pi}{3}\right) = -1$, y has a minimum value of -2, at $x = \frac{7}{6}\pi$.

(2) $y = -\cos x + \sqrt{3}\sin x - 2$

[Sol] Since $a = \sqrt{3}$ and $b = -1$, $\alpha = -\frac{\pi}{6}$.

$$\therefore\ y = -\cos x + \sqrt{3}\sin x - 2 = 2\sin\left(x - \frac{\pi}{6}\right) - 2$$

From the condition, $-\frac{\pi}{6} \le x - \frac{\pi}{6} < \frac{11}{6}\pi$

Since $0 \le x < 2\pi$, $-1 \le \sin\left(x - \frac{\pi}{6}\right) \le 1$

When $\sin\left(x - \frac{\pi}{6}\right) = 1$, y has a maximum value of 0, at $x = \frac{2}{3}\pi$.

When $\sin\left(x - \frac{\pi}{6}\right) = -1$, y has a minimum value of -4, at $x = \frac{5}{3}\pi$.

KUMON

M 137 a

Addition Theorem 3

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

Given that $0 \leq x < 2\pi$, find the maximum and minimum values of each of the following functions, and state the value of x at which each occurs.

(1) $y = \sin x + \sin\left(\dfrac{\pi}{3} - x\right)$

[Sol] $y = \sin x + \left(\dfrac{\sqrt{3}}{2}\cos x - \dfrac{1}{2}\sin x\right)$

$= \dfrac{1}{2}\sin x + \dfrac{\sqrt{3}}{2}\cos x = \dfrac{1}{2}(\sin x + \sqrt{3}\cos x)$

Since $a = 1$ and $b = \sqrt{3}$, $\alpha = \dfrac{\pi}{3}$.

$\therefore\ y = \sin x + \sin\left(\dfrac{\pi}{3} - x\right) = \dfrac{1}{2}\left[2\sin\left(x + \dfrac{\pi}{3}\right)\right]$

$\therefore\ y = \sin\left(x + \dfrac{\pi}{3}\right)$

y has a maximum value of $\boxed{1}$, at $x = \boxed{\dfrac{\pi}{6}}$.

y has a minimum value of $\boxed{-1}$, at $x = \boxed{\dfrac{7}{6}\pi}$.

(2) $y = 2\cos 2x - \sin\left(\dfrac{\pi}{6} - 2x\right)$

[Sol] $y = 2\cos 2x - \left(\dfrac{1}{2}\cos 2x - \dfrac{\sqrt{3}}{2}\sin 2x\right)$

$= \dfrac{3}{2}\cos 2x + \dfrac{\sqrt{3}}{2}\sin 2x = \dfrac{\sqrt{3}}{2}(\sin 2x + \sqrt{3}\cos 2x)$

Since $a = 1$ and $b = \sqrt{3}$, $\alpha = \dfrac{\pi}{3}$.

$\therefore\ y = 2\cos 2x - \sin\left(\dfrac{\pi}{6} - 2x\right) = \dfrac{\sqrt{3}}{2}\left[2\sin\left(2x + \dfrac{\pi}{3}\right)\right]$

$\therefore\ y = \sqrt{3}\sin\left(2x + \dfrac{\pi}{3}\right)$

y has a maximum value of $\sqrt{3}$, at $x = \dfrac{\pi}{12}$ or $\dfrac{13}{12}\pi$.

y has a minimum value of $-\sqrt{3}$, at $x = \dfrac{7}{12}\pi$ or $\dfrac{19}{12}\pi$.

(3) $y = \sin x \sin\left(\frac{\pi}{3} - x\right)$

[Sol] $y = -\frac{1}{2}\left[\cos\frac{\pi}{3} - \cos\left(2x - \frac{\pi}{3}\right)\right]$

$= \frac{1}{2}\cos\left(2x - \frac{\pi}{3}\right) - \frac{1}{4}$

From the condition, $-\frac{\pi}{3} \leq 2x - \frac{\pi}{3} < \frac{11}{3}\pi$.

Since $0 \leq x < 2\pi$, $-1 \leq \cos\left(2x - \frac{\pi}{3}\right) \leq 1$

y has a maximum value of $\frac{1}{4}$, at $x = \frac{\pi}{6}$ or $\frac{7}{6}\pi$.

y has a minimum value of $-\frac{3}{4}$, at $x = \frac{2}{3}\pi$ or $\frac{5}{3}\pi$.

(4) $y = \cos\left(\frac{\pi}{4} - x\right)\cos x + \frac{3\sqrt{2}}{4}$

[Sol] $y = \frac{1}{2}\left[\cos\frac{\pi}{4} + \cos\left(\frac{\pi}{4} - 2x\right)\right] + \frac{3\sqrt{2}}{4}$

$= \frac{\sqrt{2}}{4} + \frac{1}{2}\cos\left(\frac{\pi}{4} - 2x\right) + \frac{3\sqrt{2}}{4}$

$= \sqrt{2} + \frac{1}{2}\cos\left(\frac{\pi}{4} - 2x\right) = \sqrt{2} + \frac{1}{2}\cos\left(2x - \frac{\pi}{4}\right)$

From the condition, $-\frac{\pi}{4} \leq 2x - \frac{\pi}{4} < \frac{15}{4}\pi$

Since $0 \leq x < 2\pi$, $-1 \leq \cos\left(2x - \frac{\pi}{4}\right) \leq 1$

y has a maximum value of $\sqrt{2} + \frac{1}{2}$, at $x = \frac{\pi}{8}$ or $\frac{9}{8}\pi$.

y has a minimum value of $\sqrt{2} - \frac{1}{2}$, at $x = \frac{5}{8}\pi$ or $\frac{13}{8}\pi$.

Addition Theorem 3

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

Given that $0 \leq x < 2\pi$, find the maximum and minimum values of each of the following functions, and state the value of x at which each occurs.

(1) $y = 2\sin x - \cos 2x$

[Sol] $y = 2\sin x - (1 - 2\sin^2 x) = 2\sin^2 x + 2\sin x - 1$

Letting $X = \sin x$, $\quad -1 \leq X \leq 1$

$\therefore\ y = 2X^2 + 2X - 1 = 2\left(X + \frac{1}{2}\right)^2 - \frac{3}{2}$

$\therefore$ When $X = 1$, the function has a maximum value of 3, at $x = \frac{\pi}{2}$.

When $X = -\frac{1}{2}$, the function has a minimum value of $-\frac{3}{2}$, at $x = \frac{7}{6}\pi, \frac{11}{6}\pi$.

(2) $y = 4\sin x + \cos 2x$

[Sol] $y = 4\sin x + (1 - 2\sin^2 x) = -2\sin^2 x + 4\sin x + 1$

Letting $X = \sin x$, $\quad -1 \leq X \leq 1$

$\therefore\ y = -2X^2 + 4X + 1 = -2(X - 1)^2 + 3$

$\therefore$ When $X = 1$, the function has a maximum value of 3, at $x = \frac{\pi}{2}$.

When $X = -1$, the function has a minimum value of -5, at $x = \frac{3}{2}\pi$.

(3) $y = \cos 2x - 2\cos x$

[Sol] $y = (2\cos^2 x - 1) - 2\cos x$

Letting $X = \cos x$, $\quad -1 \leq X \leq 1$

$\therefore\ y = 2X^2 - 2X - 1 = 2\left(X - \frac{1}{2}\right)^2 - \frac{3}{2}$

$\therefore$ When $X = -1$, the function has a maximum value of 3, at $x = \pi$.

When $X = \frac{1}{2}$, the function has a minimum value of $-\frac{3}{2}$, at $x = \frac{\pi}{3}, \frac{5}{3}\pi$.

M 138 b

(4) $y = -\frac{1}{2}\cos 2x - \sqrt{2}\sin x - \frac{5}{2}$

[Sol] $y = -\frac{1}{2}(1 - 2\sin^2 x) - \sqrt{2}\sin x - \frac{5}{2}$

$= -\frac{1}{2} + \sin^2 x - \sqrt{2}\sin x - \frac{5}{2} = \sin^2 x - \sqrt{2}\sin x - 3$

Letting $X = \sin x$, $\quad -1 \leq X \leq 1$

$= X^2 - \sqrt{2}X - 3$

$= \left(X - \frac{\sqrt{2}}{2}\right)^2 - \frac{7}{2}$

$\therefore$ When $X = -1$, the function has a maximum value of $\sqrt{2} - 2$, at $x = \frac{3}{2}\pi$.

When $X = \frac{\sqrt{2}}{2}$, the function has a minimum value of $-\frac{7}{2}$, at $x = \frac{\pi}{4}, \frac{3}{4}\pi$.

(5) $y = \cos 2x + 2\sqrt{3}\sin x - \frac{3}{2}$

[Sol] $y = (1 - 2\sin^2 x) + 2\sqrt{3}\sin x - \frac{3}{2}$

$= -2\sin^2 x + 2\sqrt{3}\sin x - \frac{1}{2}$

Letting $X = \sin x$, $\quad -1 \leq X \leq 1$

$= -2X^2 + 2\sqrt{3}X - \frac{1}{2}$

$= -2\left(X - \frac{\sqrt{3}}{2}\right)^2 + 1$

$\therefore$ When $X = \frac{\sqrt{3}}{2}$, the function has a maximum value of 1, at $x = \frac{\pi}{3}, \frac{2}{3}\pi$.

When $X = -1$, the function has a minimum value of $-2\sqrt{3} - \frac{5}{2}$, at $x = \frac{3}{2}\pi$.

KUMON
M 139 a

Addition Theorem 3

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

Given that $0 \leq x < 2\pi$, find the maximum and minimum values of each of the following functions, and state the value of x at which each occurs.

(1) $y = \cos x + 2\cos\frac{x}{2}$ **(Hint)** $\cos x = \cos\left(2 \cdot \frac{x}{2}\right)$

[Sol] $y = \cos\left(2 \cdot \frac{x}{2}\right) + 2\cos\frac{x}{2}$

$= 2\cos^2\frac{x}{2} - 1 + 2\cos\frac{x}{2}$

$= 2\cos^2\frac{x}{2} + 2\cos\frac{x}{2} - 1$

Letting $X = \cos\frac{x}{2}$, $-1 < X \leq 1$

$= 2X^2 + 2X - 1$

$= 2\left(X + \frac{1}{2}\right)^2 - \frac{3}{2}$

$\therefore$ When $X = 1$, the function has a maximum value of 3, at $x = 0$.

When $X = -\frac{1}{2}$, the function has a minimum value of $-\frac{3}{2}$, at $x = \frac{4}{3}\pi$.

(2) $y = \cos 2x - 2\sin x$

[Sol] $y = (1 - 2\sin^2 x) - 2\sin x$

$= -2\sin^2 x - 2\sin x + 1$

Letting $X = \sin x$, $-1 \leq X \leq 1$

$= -2X^2 - 2X + 1$

$= -2\left(X + \frac{1}{2}\right)^2 + \frac{3}{2}$

$\therefore$ When $X = -\frac{1}{2}$, the function has a maximum value of $\frac{3}{2}$, at $x = \frac{7}{6}\pi, \frac{11}{6}\pi$.

When $X = 1$, the function has a minimum value of -3, at $x = \frac{\pi}{2}$.

M 139 b

(3) $y = 2\sqrt{3}\sin x - \cos 2x$

[Sol] $y = 2\sqrt{3}\sin x - (1 - 2\sin^2 x)$

$= 2\sin^2 x + 2\sqrt{3}\sin x - 1$

Letting $X = \sin x, \quad -1 \leq X \leq 1$

$= 2X^2 + 2\sqrt{3}X - 1$

$= 2\left(X + \frac{\sqrt{3}}{2}\right)^2 - \frac{5}{2}$

$\therefore$ When $X = 1$, the function has a maximum value of $2\sqrt{3} + 1$, at $x = \frac{\pi}{2}$.

When $X = -\frac{\sqrt{3}}{2}$, the function has a minimum value of $-\frac{5}{2}$, at $x = \frac{4}{3}\pi, \frac{5}{3}\pi$.

(4) $y = \cos 4x - 4\cos 2x + 3$

[Sol] $y = \cos(2 \cdot 2x) - 4\cos 2x + 3$

$= (2\cos^2 2x - 1) - 4\cos 2x + 3$

$= 2\cos^2 2x - 4\cos 2x + 2$

Letting $X = \cos 2x, \quad -1 \leq X \leq 1$

$= 2X^2 - 4X + 2$

$= 2(X - 1)^2$

$\therefore$ When $X = -1$, the function has a maximum value of 8, at $x = \frac{\pi}{2}, \frac{3}{2}\pi$.

When $X = 1$, the function has a minimum value of 0, at $x = 0, \pi$.

M 140 a

Addition Theorem 3

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. Convert each of the following expressions into the form $r\sin(kx+\alpha)$, $(-\pi<\alpha<\pi)$.

(1) $\sqrt{3}\sin\theta+\cos\theta$

[Sol] Since $a=\sqrt{3}$ and $b=1$, $\alpha=\dfrac{\pi}{6}$

$$\therefore\ \sqrt{3}\sin\theta+\cos\theta=2\sin\left(\theta+\frac{\pi}{6}\right)$$

(2) $2\sin\theta-2\cos\theta$

[Sol] Since $a=2$ and $b=-2$, $\alpha=-\dfrac{\pi}{4}$

$$\therefore\ 2\sin\theta-2\cos\theta=2\sqrt{2}\sin\left(\theta-\frac{\pi}{4}\right)$$

2. Solve each of the following equations. $(0\leq x<2\pi)$

(1) $\sin x+\cos x=-1$

[Sol] Since $a=1$ and $b=1$, $\alpha=\dfrac{\pi}{4}$

The new equation: $\sqrt{2}\sin\left(x+\dfrac{\pi}{4}\right)=-1$

$$\sin\left(x+\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$$

From the condition, $\dfrac{\pi}{4}\leq x+\dfrac{\pi}{4}<\dfrac{9}{4}\pi$

$$\therefore\ x=\pi,\frac{3}{2}\pi$$

(2) $\sin x-\sqrt{3}\cos x=1$

[Sol] Since $a=1$ and $b=-\sqrt{3}$, $\alpha=-\dfrac{\pi}{3}$

The new equation: $2\sin\left(x-\dfrac{\pi}{3}\right)=1$

$$\sin\left(x-\frac{\pi}{3}\right)=\frac{1}{2}$$

From the condition, $-\dfrac{\pi}{3}\leq x-\dfrac{\pi}{3}<\dfrac{5}{3}\pi$

$$\therefore\ x=\frac{\pi}{2},\frac{7}{6}\pi$$

3. Solve the following inequality. $(0 \le x < 2\pi)$

$\sin x \ge -\cos 2x$

[Sol] $\sin x + \cos 2x \ge 0$

$\sin x + (1 - 2\sin^2 x) \ge 0$

$2\sin^2 x - \sin x - 1 \le 0$

$(2\sin x + 1)(\sin x - 1) \le 0$

$-\frac{1}{2} \le \sin x \le 1$

$\therefore\ 0 \le x \le \frac{7}{6}\pi,\ \frac{11}{6}\pi \le x < 2\pi$

4. Given that $0 \le x < 2\pi$, find the maximum and minimum value of $y = \sin x - \cos x - \frac{1}{2}$ and state the value of x at which each occurs.

[Sol] Since $a = 1$ and $b = -1$, $\alpha = -\frac{\pi}{4}$

$\therefore\ y = \sin x - \cos x - \frac{1}{2} = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right) - \frac{1}{2}$

From the condition, $-\frac{\pi}{4} \le x - \frac{\pi}{4} \le \frac{7}{4}\pi$

Since $0 \le x < 2\pi$, $\quad -1 \le \sin\left(x - \frac{\pi}{4}\right) \le 1$

$\therefore$ When $\sin\left(x - \frac{\pi}{4}\right) = 1$, the function has a maximum value of $\sqrt{2} - \frac{1}{2}$, at $x = \frac{3}{4}\pi$.

When $\sin\left(x - \frac{\pi}{4}\right) = -1$, the function has a minimum value of $-\sqrt{2} - \frac{1}{2}$, at $x = \frac{7}{4}\pi$.

M 141 a

Coordinates of a Point

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

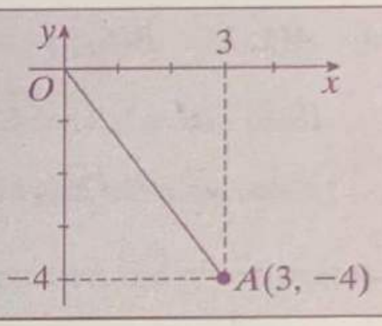
1. In each of the following exercises, use the Pythagorean Theorem to find the distance between the two given points.

Ex. $O(0, 0)$, $A(3, -4)$

[Sol] $OA = \sqrt{(3-0)^2 + (-4-0)^2}$

$= \sqrt{25}$

$= 5$



(1) $O(0, 0)$, $A(6, -8)$

[Sol] $OA = \sqrt{(6-0)^2 + (-8-0)^2}$

$= \sqrt{100}$

$= 10$

(2) $O(0, 0)$, $A(4, -4)$

[Sol] $OA = \sqrt{(4-0)^2 + (-4-0)^2}$

$= \sqrt{32}$

$= 4\sqrt{2}$

(3) $A(2, 0)$, $B(1, -5)$

[Sol] $AB = \sqrt{(1-2)^2 + (-5-0)^2}$

$= \sqrt{26}$

(4) $A(0, 4)$, $B(7, -3)$

[Sol] $AB = \sqrt{(7-0)^2 + (-3-4)^2}$

$= \sqrt{98}$

$= 7\sqrt{2}$

(5) $A(0, 3)$, $B(-4, 0)$

[Sol] $AB = \sqrt{(-4-0)^2 + (0-3)^2}$

$= \sqrt{25}$

$= 5$

(6) $A(1, 2)$, $B(5, 5)$

[Sol] $AB = \sqrt{(5-1)^2 + (5-2)^2}$

$= \sqrt{25}$

$= 5$

(7) $A(-7, -4)$, $B(-2, -3)$

[Sol] $AB = \sqrt{(-2+7)^2 + (-3+4)^2}$

$= \sqrt{26}$

2. In each exercise below, find the lengths of all three sides of the triangle formed by the given points.

(1) $O(0, 0)$, $A(3, -4)$, $B(-4, 3)$

[Sol] $OA = \sqrt{(3-0)^2 + (-4-0)^2} = 5$

$OB = \sqrt{(-4-0)^2 + (3-0)^2} = 5$

$AB = \sqrt{(-4-3)^2 + (3+4)^2} = 7\sqrt{2}$

(2) $A(-5, 0)$, $B(-1, -7)$, $C(2, 4)$

[Sol] $AB = \sqrt{(-1+5)^2 + (-7-0)^2} = \sqrt{65}$

$AC = \sqrt{(2+5)^2 + (4-0)^2} = \sqrt{65}$

$BC = \sqrt{(2+1)^2 + (4+7)^2} = \sqrt{130}$

Coordinates of a Point

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

Distance Formula

The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is expressed as:

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

[**Note**: Using $(x_1 - x_2)^2$ and $(y_1 - y_2)^2$ will give the same result.]

1. Use the Distance Formula to find the distance between the given points.

(1) $O(0, 0)$, $A(3, 4)$

[Sol] $OA = \sqrt{(3-0)^2 + (4-0)^2} = \sqrt{25} = 5$

(2) $A(2, 1)$, $B(5, -2)$

[Sol] $AB = \sqrt{(5-2)^2 + (-2-1)^2} = \sqrt{18} = 3\sqrt{2}$

(3) $A(-1, 3)$, $B(-2, -6)$

[Sol] $AB = \sqrt{(-2+1)^2 + (-6-3)^2} = \sqrt{82}$

(4) $A(8, 2)$, $B(-1, 4)$

[Sol] $AB = \sqrt{(-1-8)^2 + (4-2)^2} = \sqrt{85}$

2. Find the lengths of the diagonals of the quadrilateral formed by the given points below.

$$A(3, 2), \quad B(-1, 3), \quad C(0, 0), \quad D(4, 0)$$

[Sol] $AC = \sqrt{(0-3)^2 + (0-2)^2} = \sqrt{13}$

$BD = \sqrt{(4+1)^2 + (0-3)^2} = \sqrt{34}$

3. Given that point $Q(9, 2)$ is 13 units away from point P, and that the y-coordinate of P is 7, find the x-coordinate of point P.

[Sol] Letting the coordinates of point P be $(x, 7)$, and since $PQ = 13$,

$$\sqrt{(x-9)^2 + (\boxed{7-2})^2} = 13$$

$$x^2 - 18x - \boxed{63} = 0$$

$$(x-21)(x+3) = 0$$

$$x = 21, \quad -3$$

4. Given that point $R(4, 1)$ is $3\sqrt{5}$ units away from point S, and that the x-coordinate of S is -2, find the y-coordinate of point S.

[Sol] Letting the coordinates of point S be $(-2, y)$, and since $RS = 3\sqrt{5}$,

$$\sqrt{(-2-4)^2 + (y-1)^2} = 3\sqrt{5}$$

$$y^2 - 2y - 8 = 0$$

$$(y-4)(y+2) = 0$$

$$y = 4, \quad -2$$

KUMON

M 143 a　Coordinates of a Point

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	1	2	3	4~

1. Given that M is the midpoint of segment AB, in each of the following exercises, prove that $AM = MB$.

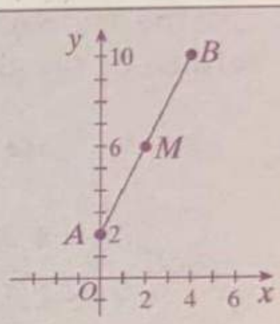
Ex. $A(0, 2)$, $M(2, 6)$, $B(4, 10)$

[Sol] Using the distance formula,

$$AM = \sqrt{(2-0)^2 + (6-2)^2} = \sqrt{20} = 2\sqrt{5}$$

$$MB = \sqrt{(4-2)^2 + (10-6)^2} = \sqrt{20} = 2\sqrt{5}$$

Therefore, $AM = MB$



Reference

(1) $A(0, -6)$, $M(-1, -1)$, $B(-2, 4)$

[Sol] Using the distance formula,

$$AM = \sqrt{(-1-0)^2 + (-1+6)^2} = \sqrt{26}$$

$$MB = \sqrt{(-2+1)^2 + (4+1)^2} = \sqrt{26}$$

Therefore, $AM = MB$

(2) $A(-1, -2)$, $M(1, 0)$, $B(3, 2)$

[Sol] Using the distance formula,

$$AM = \sqrt{(1+1)^2 + (0+2)^2} = 2\sqrt{2}$$

$$MB = \sqrt{(3-1)^2 + (2-0)^2} = 2\sqrt{2}$$

Therefore, $AM = MB$

2. Given that point $M(a, b)$ is the midpoint of segment AB, (where AB has coordinates $A(1, 0)$, $B(3, 6)$ and $AB = 2\sqrt{10}$), find the values of a and b.

[Sol] $AM = \boxed{\sqrt{(a-1)^2 + b^2}} = \sqrt{10}$ . . . ①

$MB = \sqrt{(3-a)^2 + (6-b)^2} = \sqrt{10}$ . . . ②

Simplifying, ① becomes $a^2 - 2a + 1 + b^2 = 10$ . . . ③

② becomes $a^2 - 6a + 9 + b^2 - 12b + 36 = 10$ . . . ④

From ③-④, $4a + 12b - 44 = 0$

$a = -3b + 11$ . . . ⑤

Substituting ⑤ into ③,

$b = 3$

Calculation: $(-3b + 11)^2 - 2(-3b + 11) + b^2 - 9 = 0$

$9b^2 - 66b + 121 + 6b - 22 + b^2 - 9 = 0$

$10b^2 - 60b + 90 = 0$

$b^2 - 6b + 9 = 0$

$(b-3)^2 = 0$

Substituting this value into ⑤,

$a = 2$

Therefore, $a = 2$, $b = 3$

Midpoint Formula

The coordinates of the midpoint of a line segment connecting two points (x_1, y_1) and (x_2, y_2) are:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

3. Use the Midpoint Formula to obtain the midpoint of the line segment formed by the given points.

(1) $O(0, 0),\quad A(4, 6)$

[Sol] $\left(\frac{0+4}{2}, \frac{0+6}{2}\right)$

$\therefore (2, 3)$

(2) $A(4, 0),\quad B(6, 2)$

[Sol] $\left(\frac{4+6}{2}, \frac{0+2}{2}\right)$

$\therefore (5, 1)$

(3) $A(-2, -4),\quad B(1, 7)$

[Sol] $\left(\frac{-2+1}{2}, \frac{-4+7}{2}\right)$

$\therefore \left(-\frac{1}{2}, \frac{3}{2}\right)$

(4) $A(1, 5),\quad B(4, 8)$

[Sol] $\left(\frac{1+4}{2}, \frac{5+8}{2}\right)$

$\therefore \left(\frac{5}{2}, \frac{13}{2}\right)$

(5) $A(3, -7),\quad B(-5, 3)$

[Sol] $\left(\frac{3-5}{2}, \frac{-7+3}{2}\right)$

$\therefore (-1, -2)$

(6) $A(-1, -5),\quad B(3, -1)$

[Sol] $\left(\frac{-1+3}{2}, \frac{-5-1}{2}\right)$

$\therefore (1, -3)$

(7) $A(-3, 5),\quad B(-1, 2)$

[Sol] $\left(\frac{-3-1}{2}, \frac{5+2}{2}\right)$

$\therefore \left(-2, \frac{7}{2}\right)$

(8) $A(-5, -7),\quad B(-3, 12)$

[Sol] $\left(\frac{-5-3}{2}, \frac{-7+12}{2}\right)$

$\therefore \left(-4, \frac{5}{2}\right)$

Coordinates of a Point

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

Ex. Given that point $M(5, 2)$ is the midpoint of segment AB, where the coordinates of point A are $(3, 5)$, find the coordinates of point B.

[Sol] Letting (x_2, y_2) be the coordinates of point B,

Using the Midpoint Formula,

$$\frac{3+x_2}{2}=5 \quad \text{and} \quad \frac{\boxed{5}+y_2}{2}=\boxed{2}$$

Solving, $x_2=\boxed{7}$ $y_2=\boxed{-1}$

Therefore, the coordinates of B are $(\boxed{7}, \boxed{-1})$.

Answers: 5, 2, 7, −1, 7, −1

1. Given that point $M(-2, 5)$ is the midpoint of segment AB, where the coordinates of point A are $(-5, 2)$, find the coordinates of point B.

[Sol] Letting (x_2, y_2) be the coordinates of point B,

$$\frac{-5+x_2}{2}=-2 \quad \text{and} \quad \frac{2+y_2}{2}=5$$

Solving, $x_2=1$ $y_2=8$

Therefore, the coordinates of B are $(1, 8)$.

2. Given that point $M(-1, -3)$ is the midpoint of segment AB, where the coordinates of point A are $(-5, -8)$, find the coordinates of point B.

[Sol] Letting (x_2, y_2) be the coordinates of point B,

$$\frac{-5+x_2}{2}=-1 \quad \text{and} \quad \frac{-8+y_2}{2}=-3$$

Solving, $x_2=3$ $y_2=2$

Therefore, the coordinates of B are $(3, 2)$.

3. Given that point $M\left(\frac{1}{2}, -2\right)$ is the midpoint of segment AB, where the coordinates of point B are $(5, -6)$, find the coordinates of point A.

[Sol] Letting (x_1, y_1) be the coordinates of point A,

$$\frac{x_1+5}{2}=\frac{1}{2} \quad \text{and} \quad \frac{y_1-6}{2}=-2$$

Solving, $\quad x_1 = -4 \qquad y_1 = 2$

Therefore, the coordinates of A are $(-4, 2)$.

4. Given $\triangle ABC$, with vertices $A(5, 2)$, $B(-2, -3)$ and $C(8, -5)$, find the midpoint of each side of the triangle.

[Sol]

Midpoint of side AB: $\left(\frac{5-2}{2}, \frac{2-3}{2}\right) \qquad \therefore \left(\frac{3}{2}, -\frac{1}{2}\right)$

Midpoint of side BC: $\left(\frac{-2+8}{2}, \frac{-3-5}{2}\right) \qquad \therefore (3, -4)$

Midpoint of side CA: $\left(\frac{8+5}{2}, \frac{-5+2}{2}\right) \qquad \therefore \left(\frac{13}{2}, -\frac{3}{2}\right)$

KUMON

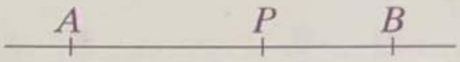
M 145 a

Coordinates of a Point

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

Given that a point P lies on segment AB, the line segment AB is said to be *internally divided by P* into segments AP and PB.



Note: The notation is specific to the direction of the segments.
Line segment BA is *internally divided by P* into segments BP and PA.

Complete and check your answers.

Line segment QR is *internally divided by C* into segments $\boxed{QC}$ and $\boxed{CR}$.

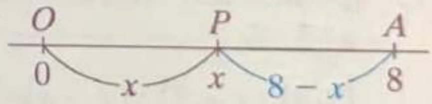
Line segment DC is *internally divided by M* into segments $\boxed{DM}$ and $\boxed{MC}$.

Line segment GH is *internally divided by K* into segments $\boxed{GK}$ and $\boxed{KH}$.

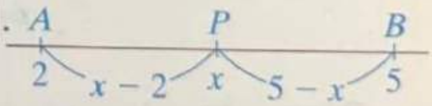
Answers: QC, CR, DM, MC, GK, KH

1. In each of the following exercises, complete the diagram and the given statement.

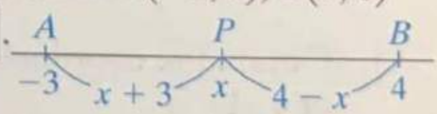
(1) If OA is internally divided by P, given the coordinates $O(0, 0)$, $A(8, 0)$ and $P(x, 0)$, $OP = x$ and $PA = \boxed{8 - x}$.



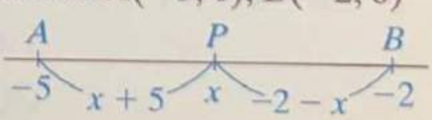
(2) If AB is internally divided by P, given the coordinates $A(2, 0)$, $B(5, 0)$ and $P(x, 0)$, $AP = \boxed{x - 2}$ and $PB = \boxed{5 - x}$.



(3) If AB is internally divided by P, given the coordinates $A(-3, 0)$, $B(4, 0)$ and $P(x, 0)$, $AP = \boxed{x + 3}$ and $PB = \boxed{4 - x}$.



(4) If AB is internally divided by P, given the coordinates $A(-5, 0)$, $B(-2, 0)$ and $P(x, 0)$, $AP = \boxed{x + 5}$ and $PB = \boxed{-2 - x}$.



2. Given points $A(3, 0)$ and $B(7, 0)$:

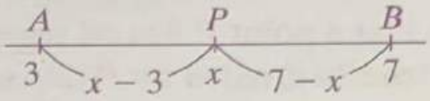
(1) Obtain the coordinates of point P internally dividing AB in the ratio of $3 : 2$.

[Sol] $(x-3):(7-x)=3:2$

$$3(7-x)=2(x-3)$$

$$x=\frac{27}{5}$$

$$\therefore P\left(\boxed{\frac{27}{5}}, \boxed{0}\right)$$



(2) Obtain the coordinates of point Q internally dividing BA in the ratio of $3 : 2$.

[Sol] $(7-x):(x-3)=3:2$

$$x=\frac{23}{5}$$

$$\therefore Q\left(\frac{23}{5}, 0\right)$$

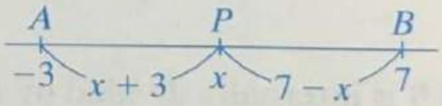
3. Given points $A(-3, 0)$ and $B(7, 0)$:

(1) Obtain the coordinates of point P internally dividing AB in the ratio of $3 : 2$.

[Sol] $(x+3):(7-x)=3:2$

$$x=3$$

$$\therefore P(3, 0)$$

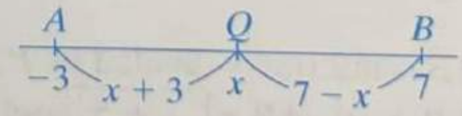


(2) Obtain the coordinates of point Q internally dividing BA in the ratio of $4 : 1$.

[Sol] $(7-x):(x+3)=4:1$

$$x=-1$$

$$\therefore Q(-1, 0)$$



KUMON

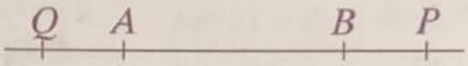
M 146 a

Coordinates of a Point

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

Given that a point P lies on the same line as segment AB, but outside of segment AB, AB is said to be *externally divided by P* into AP and PB.



1. Given the points Q, A, B and P in that order on a line, complete the following statements.

(1) Line segment BA is *externally divided by P* into $\boxed{BP}$ and $\boxed{PA}$.

(2) Line segment AB is *externally divided by Q* into $\boxed{AQ}$ and $\boxed{QB}$.

(3) Line segment BA is *externally divided by Q* into $\boxed{BQ}$ and $\boxed{QA}$.

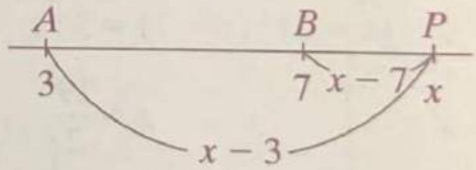
2. Given points $A(3, 0)$ and $B(7, 0)$, obtain the coordinates of point P externally dividing AB in the ratio of $3:1$.

[Sol] $(x-3):(x-7) = 3:1$

$3(x-7) = (x-3)$

$x = 9$

$P(9, 0)$



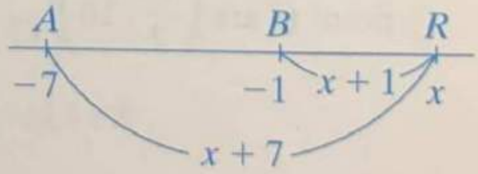
3. Given points $A(-7, 0)$ and $B(-1, 0)$, obtain the coordinates of point R externally dividing BA in the ratio of $2:5$.

[Sol] $(x+1):(x+7) = 2:5$

$2(x+7) = 5(x+1)$

$x = 3$

$R(3, 0)$

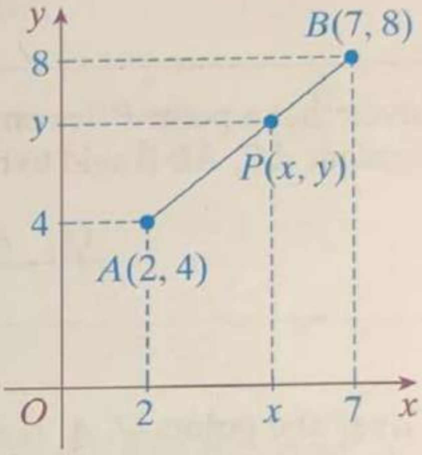


4. Given points $A(2, 4)$ and $B(7, 8)$, obtain the coordinates of point P *internally dividing* line segment AB in the ratio of $3 : 2$.

[Sol] Letting P's coordinates be (x, y),

$$(x - 2) : (7 - x) = 3 : 2$$
$$x = 5$$
$$(y - 4) : (8 - y) = 3 : 2$$
$$y = \frac{32}{5}$$

Therefore, the coordinates of point P are $\left(5, \frac{32}{5}\right)$.

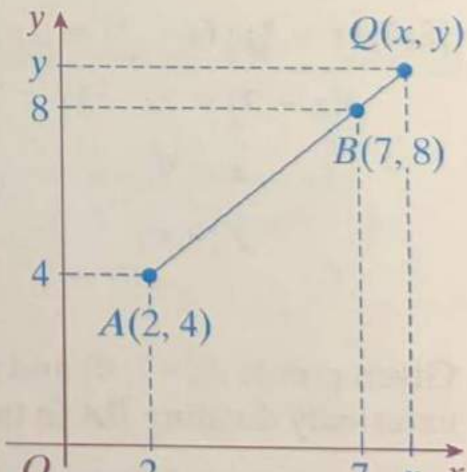


5. Obtain the coordinates of point Q externally dividing AB (from exercise 4., above) in the ratio of $3 : 1$. $[AQ : QB = 3 : 1]$.

[Sol] Letting Q be $Q(x, y)$,

$$(x - 2) : (x - 7) = 3 : 1$$
$$x = \frac{19}{2}$$
$$(y - 4) : (y - 8) = 3 : 1$$
$$y = 10$$

Therefore, the coordinates of point Q are $\left(\frac{19}{2}, 10\right)$.



Coordinates of a Point

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

The coordinates of the points which divide the line segment connecting points $A(x_1, y_1)$ and $B(x_2, y_2)$ are expressed as follows:

Internally in the ratio $m:n$, $\left(\dfrac{mx_2+nx_1}{m+n}, \dfrac{my_2+ny_1}{m+n}\right)$

Externally in the ratio $m:n$, $\left(\dfrac{mx_2-nx_1}{m-n}, \dfrac{my_2-ny_1}{m-n}\right)$

1. Obtain the coordinates of each of the following points which divide the line segment that connects the points $A(-3, 2)$ and $B(4, -8)$, and then graph the points.

(1) Point C dividing it into two equal segments.

[Sol] $\dfrac{-3+4}{2}=\dfrac{1}{2}$

$\dfrac{2-8}{2}=-3 \qquad \therefore C\left(\dfrac{1}{2}, -3\right)$

(2) Point D internally dividing it in the ratio of 3 : 1.

[Sol] $\dfrac{3\cdot4+1\cdot(-3)}{3+1}=\dfrac{9}{4}$

$\dfrac{3\cdot(-8)+1\cdot2}{3+1}=-\dfrac{11}{2} \quad \therefore D\left(\dfrac{9}{4}, -\dfrac{11}{2}\right)$

(3) Point E externally dividing it in the ratio of 3 : 1.

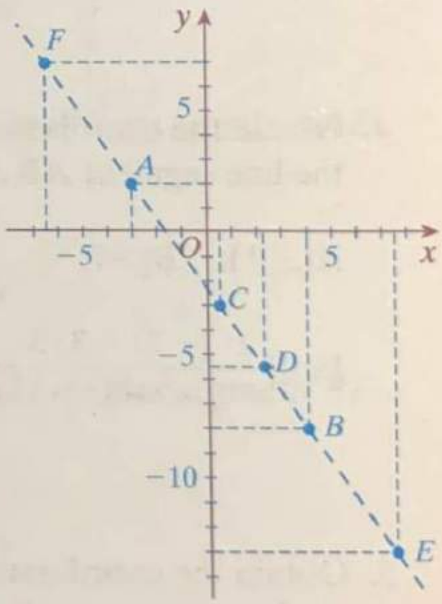
[Sol] $\dfrac{3\cdot4-1\cdot(-3)}{3-1}=\dfrac{15}{2}$

$\dfrac{3\cdot(-8)-1\cdot2}{3-1}=-13 \quad \therefore E\left(\dfrac{15}{2}, -13\right)$

(4) Point F externally dividing it in the ratio of 1 : 3.

[Sol] $\dfrac{1\cdot4-3\cdot(-3)}{1-3}=-\dfrac{13}{2}$

$\dfrac{1\cdot(-8)-3\cdot2}{1-3}=7 \quad \therefore F\left(-\dfrac{13}{2}, 7\right)$



2. Obtain the coordinates of the point internally dividing in the ratio of $2:3$ the line segment AB connecting the following pair of points.

$A(3, 8)$, $B(5, 6)$

[Sol] $\dfrac{2\cdot 5+3\cdot 3}{2+3}=\dfrac{19}{5}$, $\dfrac{2\cdot 6+3\cdot 8}{2+3}=\dfrac{36}{5}$ Ans. $\left(\dfrac{19}{5}, \dfrac{36}{5}\right)$

3. Obtain the coordinates of the point internally dividing in the ratio of $3:4$ the line segment AB connecting the following pair of points.

$A(-1, 1)$, $B(2, 8)$

[Sol] $\dfrac{3\cdot 2+4\cdot(-1)}{3+4}=\dfrac{2}{7}$, $\dfrac{3\cdot 8+4\cdot 1}{3+4}=4$ Ans. $\left(\dfrac{2}{7}, 4\right)$

4. Obtain the coordinates of the point externally dividing in the ratio of $2:3$ the line segment AB connecting the following pair of points.

$A(3, 1)$, $B(-7, 5)$

[Sol] $\dfrac{2\cdot(-7)-3\cdot 3}{2-3}=23$, $\dfrac{2\cdot 5-3\cdot 1}{2-3}=-7$ Ans. $(23, -7)$

5. Obtain the coordinates of the point externally dividing in the ratio of $5:1$ the line segment AB connecting the following pair of points.

$A(6, -4)$, $B(-2, 2)$

[Sol] $\dfrac{5\cdot(-2)-1\cdot 6}{5-1}=-4$, $\dfrac{5\cdot 2-1\cdot(-4)}{5-1}=\dfrac{7}{2}$ Ans. $\left(-4, \dfrac{7}{2}\right)$

M 148 a

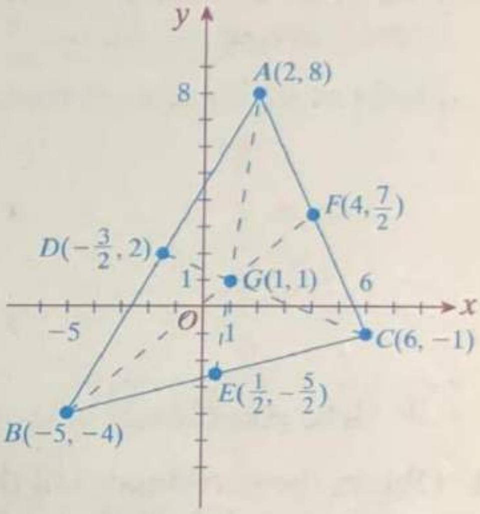
Coordinates of a Point

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

1. Given $\triangle ABC$, with vertices $A(2, 8)$, $B(-5, -4)$ and $C(6, -1)$, solve the following exercises.

(1) Find the coordinates of points D, E and F which are the midpoints of sides AB, BC and CA respectively.



[Sol] D: $\left(\dfrac{2-5}{2}, \dfrac{8-4}{2}\right)$ $\therefore D\left(-\dfrac{3}{2}, 2\right)$

E: $\left(\dfrac{-5+6}{2}, \dfrac{-4-1}{2}\right)$ $\therefore E\left(\dfrac{1}{2}, -\dfrac{5}{2}\right)$

F: $\left(\dfrac{6+2}{2}, \dfrac{-1+8}{2}\right)$ $\therefore F\left(4, \dfrac{7}{2}\right)$

(2) Find the coordinates of point G_1, which internally divides segment AE in the ratio of $2:1$.

[Sol] G_1: $\left(\dfrac{2\cdot\frac{1}{2}+1\cdot2}{2+1}, \dfrac{2\left(-\frac{5}{2}\right)+1\cdot8}{2+1}\right)$ $\therefore G_1(1, 1)$

(3) Find the coordinates of point G_2, which internally divides segment BF in the ratio of $2:1$.

[Sol] G_2: $\left(\dfrac{2\cdot4+1(-5)}{2+1}, \dfrac{2\cdot\frac{7}{2}+1(-4)}{2+1}\right)$ $\therefore G_2(1, 1)$

(4) Find the coordinates of point G_3, which internally divides segment CD in the ratio of $2:1$.

[Sol] G_3: $\left(\dfrac{2\left(-\frac{3}{2}\right)+1\cdot6}{2+1}, \dfrac{2\cdot2+1(-1)}{2+1}\right)$ $\therefore G_3(1, 1)$

Note Summary:
The line segment connecting one vertex of a triangle and the midpoint of the opposite side is called a **median**. The three medians of a triangle intersect at one point, the **center of gravity,** which *internally divides* each median in the ratio of $2:1$.

Center of Gravity Formula

The coordinates of the center of gravity of $\triangle ABC$ whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are:

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

2. Obtain the coordinates of the center of gravity of $\triangle ABC$ whose three vertices are $A(-4, 8)$, $B(-7, -5)$ and $C(5, 0)$.

[Sol] Letting the coordinates of the center of gravity be (x, y),

$$x = \frac{-4 - 7 + 5}{3} = -2$$

$$y = \frac{8 - 5 + 0}{3} = 1$$

The coordinates of the center of gravity are $(-2, 1)$.

3. Obtain the coordinates of the center of gravity of $\triangle ABC$ whose three vertices are $A(2, 3)$, $B(0, -6)$ and $C(-4, 0)$.

[Sol] Letting the coordinates of the center of gravity be (x, y),

$$x = \frac{2 + 0 - 4}{3} = -\frac{2}{3}$$

$$y = \frac{3 - 6 + 0}{3} = -1$$

The coordinates of the center of gravity are $\left(-\frac{2}{3}, -1\right)$.

4. Obtain the coordinates of the vertex C of $\triangle ABC$ whose other two vertices are $A(1, 4)$ and $B(-2, -3)$, and whose center of gravity is $\left(\frac{2}{3}, 0\right)$.

[Sol] Letting C's coordinates be (x, y),

$$\frac{1 - 2 + x}{3} = \frac{2}{3} \qquad \therefore x = 3$$

$$\frac{4 - 3 + y}{3} = 0 \qquad \therefore y = -1 \qquad C(3, -1)$$

Coordinates of a Point

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1~

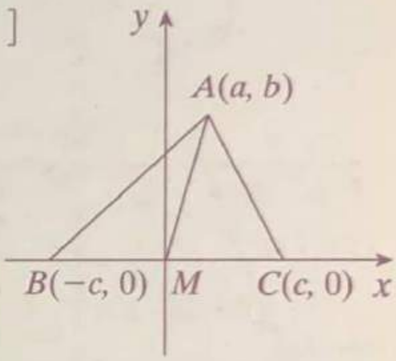
1. Given $\triangle ABC$ with M as the midpoint of side BC, prove the equality $AB^2 + AC^2 = 2(AM^2 + BM^2)$.

[Sol] Placing BC on the x-axis and M on the origin, and letting A, B and C be $A(a, b)$, $B(-c, 0)$ and $C(c, 0)$,

$$AB^2 + AC^2 = [(a+c)^2 + b^2] + [\quad (a-c)^2 + b^2 \quad]$$
$$= 2a^2 + 2b^2 + 2c^2$$

$$2(AM^2 + BM^2) = 2(a^2 + b^2 + c^2)$$

$$\therefore AB^2 + AC^2 = 2(AM^2 + BM^2)$$



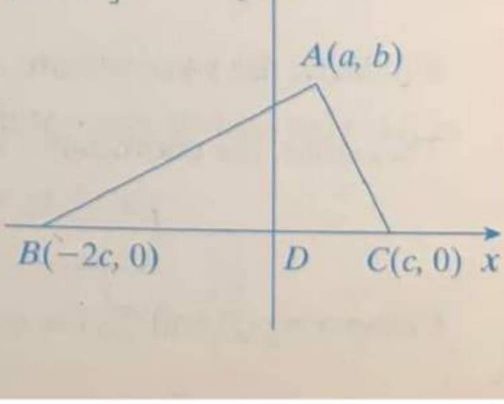
2. Given $\triangle ABC$, where point D internally divides side BC in the ratio of $2:1$, prove the following equality.

$$AB^2 + 2AC^2 = 3AD^2 + 6CD^2$$

(Hint) The origin may be placed at either B or D.

[Sol] Placing BC on the x-axis and D at the origin, and letting points A, B, and C be $A(a, b)$, $B(-2c, 0)$ and $C(c, 0)$,

$$AB^2 + 2AC^2 = (a+2c)^2 + b^2 + 2[(a-c)^2 + b^2]$$
$$= 3a^2 + 3b^2 + 6c^2$$
$$3AD^2 + 6CD^2 = 3(a^2 + b^2) + 6c^2$$
$$= 3a^2 + 3b^2 + 6c^2$$
$$\therefore AB^2 + 2AC^2 = 3AD^2 + 6CD^2$$

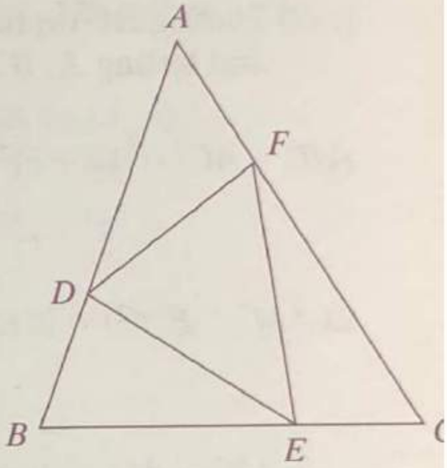


3. Let the sides AB, BC and CA of $\triangle ABC$ be internally divided in the ratio of $m : n$ by the points D, E and F, and prove that the center of gravity of $\triangle ABC$ and $\triangle DEF$ coincide.

[Sol] Let the centers of gravity of $\triangle ABC$ and $\triangle DEF$ be G and G' respectively, and the coordinates of A, B and C be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.

The coordinates of G become:

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right) \cdots ①$$



Next, obtaining the coordinates of D, E and F (the vertices of $\triangle DEF$),

$$D\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right)$$

$$E\left(\frac{mx_3 + nx_2}{m+n}, \frac{my_3 + ny_2}{m+n} \right)$$

$$F\left(\frac{mx_1 + nx_3}{m+n}, \frac{my_1 + ny_3}{m+n} \right)$$

Therefore, the x-coordinate of G' is:

$$\frac{\dfrac{mx_2 + nx_1}{m+n} + \dfrac{mx_3 + nx_2}{m+n} + \dfrac{mx_1 + nx_3}{m+n}}{3} = \frac{x_1 + x_2 + x_3}{3}$$

Similarly, the y-coordinate of G' is: $\dfrac{y_1 + y_2 + y_3}{3}$.

Therefore, the coordinates of G' are:

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right) \quad \cdots ②$$

Comparing ① and ②, the centers of gravity of $\triangle ABC$ and $\triangle DEF$ coincide.

M 150 a Coordinates of a Point

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

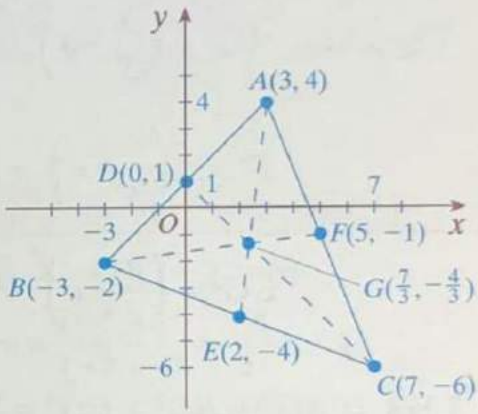
1. Given $\triangle ABC$, with vertices $A(3, 4)$, $B(-3, -2)$ and $C(7, -6)$, solve the following exercises.

(1) Find the coordinates of points D, E and F which are the midpoints of sides AB, BC and CA respectively.

[Sol] D: $\left(\dfrac{3-3}{2}, \dfrac{4-2}{2}\right)$ $\quad \therefore D(0, 1)$

E: $\left(\dfrac{-3+7}{2}, \dfrac{-2-6}{2}\right)$ $\quad \therefore E(2, -4)$

F: $\left(\dfrac{7+3}{2}, \dfrac{-6+4}{2}\right)$ $\quad \therefore F(5, -1)$



(2) Find the length of each of the triangle's medians.

[Sol] $AE = \sqrt{(2-3)^2 + (-4-4)^2} = \sqrt{65}$

$BF = \sqrt{(5+3)^2 + (-1+2)^2} = \sqrt{65}$

$CD = \sqrt{(0-7)^2 + (1+6)^2} = 7\sqrt{2}$

(3) Find the coordinates of the center of gravity.

[Sol] $\left(\dfrac{3-3+7}{3}, \dfrac{4-2-6}{3}\right)$

$\therefore \left(\dfrac{7}{3}, -\dfrac{4}{3}\right)$

(4) Obtain the coordinates of point P which internally divides side AB in the ratio of $3:2$.

[Sol] P: $\left(\dfrac{3(-3)+2\cdot 3}{3+2}, \dfrac{3(-2)+2\cdot 4}{3+2}\right)$ $\quad \therefore P\left(-\dfrac{3}{5}, \dfrac{2}{5}\right)$

2. Given that point G is the center of gravity of $\triangle ABC$ prove the following equality.

$$AG^2 + BG^2 + CG^2 = \frac{1}{3}(AB^2 + BC^2 + CA^2)$$

[Sol] Letting the coordinates of $\triangle ABC$'s vertices be $A(a, b)$, $B(-c, 0)$ and $C(c, 0)$, G's coordinates become $\left(\frac{a}{3}, \frac{b}{3}\right)$.

Therefore, $AG^2 = \left(\frac{a}{3} - a\right)^2 + \left(\frac{b}{3} - b\right)^2 = \frac{4}{9}a^2 + \frac{4}{9}b^2$

$$BG^2 = \left(\frac{a}{3} + c\right)^2 + \left(\frac{b}{3} - 0\right)^2 = \frac{a^2}{9} + \frac{2}{3}ac + c^2 + \frac{b^2}{9}$$

$$CG^2 = \left(\frac{a}{3} - c\right)^2 + \left(\frac{b}{3} - 0\right)^2 = \frac{a^2}{9} - \frac{2}{3}ac + c^2 + \frac{b^2}{9}$$

LHS $= AG^2 + BG^2 + CG^2 = \frac{2}{3}(a^2 + b^2) + 2c^2$

$AB^2 = (-c - a)^2 + (0 - b)^2 = a^2 + 2ac + c^2 + b^2$

$BC^2 = (c + c)^2 + (0 - 0)^2 = 4c^2$

$CA^2 = (a - c)^2 + (b - 0)^2 = a^2 - 2ac + c^2 + b^2$

RHS $= \frac{1}{3}(AB^2 + BC^2 + CA^2) = \frac{2}{3}(a^2 + b^2) + 2c^2$

$$\therefore\ AG^2 + BG^2 + CG^2 = \frac{1}{3}(AB^2 + BC^2 + CA^2)$$

M 151 a Equations of Straight Lines 1

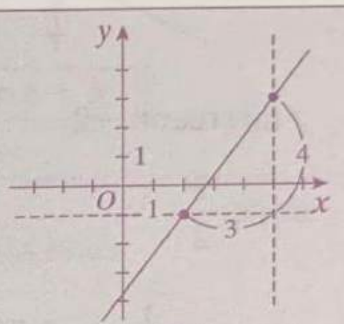
Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	1	2	3	4~

1. In each exercise below, find the slope, m, of the line that passes through the two given points.

Ex. $(5, 3),\ (2, -1)$

$$m=\frac{(-1-3)}{(2-5)}=\frac{4}{3} \longleftarrow \frac{\text{the change in } y}{\text{the change in } x}$$



[Reference]

(1) $(2, -6), (-3, 4)$

$m=\dfrac{(4+6)}{(-3-2)}=\dfrac{10}{-5}=-2$

(2) $(-1, 2), (4, -7)$

$m=\dfrac{(-7-2)}{(4+1)}=\dfrac{-9}{5}=-\dfrac{9}{5}$

(3) $(4, -3), (-1, -8)$

$m=\dfrac{(-8+3)}{(-1-4)}=\dfrac{-5}{-5}=1$

(4) $(-4, -1), (-2, -5)$

$m=\dfrac{(-5+1)}{(-2+4)}=\dfrac{-4}{2}=-2$

(5) $(6, -3), (1, -4)$

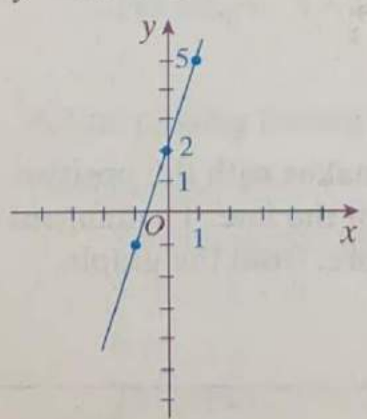
$m=\dfrac{(-4+3)}{(1-6)}=\dfrac{-1}{-5}=\dfrac{1}{5}$

(6) $(x_1, y_1), (x_2, y_2)$

$m=\dfrac{y_2-y_1}{x_2-x_1}$

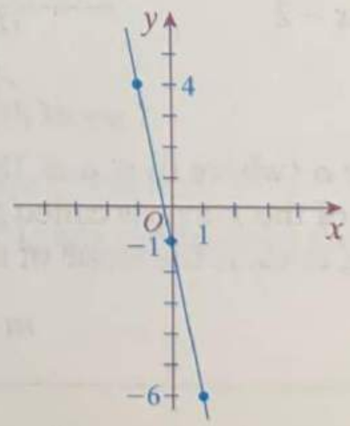
2. In each of the following exercises, draw the graph of the given linear function, and then state the y-intercept and the slope.

(1) $y = 3x + 2$



y-intercept: 2
slope: 3

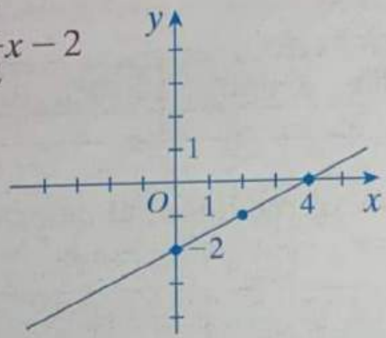
(2) $y = -5x - 1$



y-intercept: -1
slope: -5

M 151 b

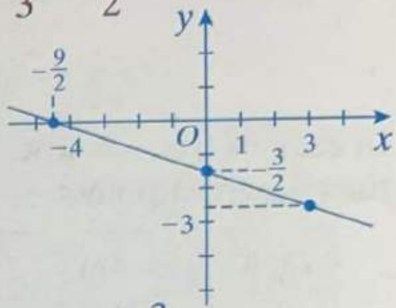
(3) $y=\frac{1}{2}x-2$



y-intercept: -2

slope: $\frac{1}{2}$

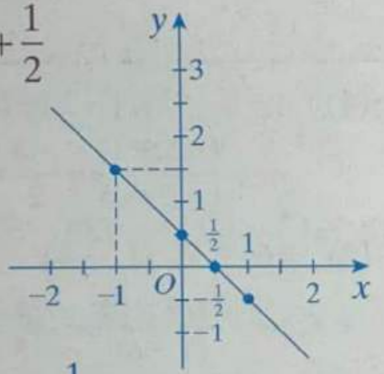
(5) $y=-\frac{1}{3}x-\frac{3}{2}$



y-intercept: $-\frac{3}{2}$

slope: $-\frac{1}{3}$

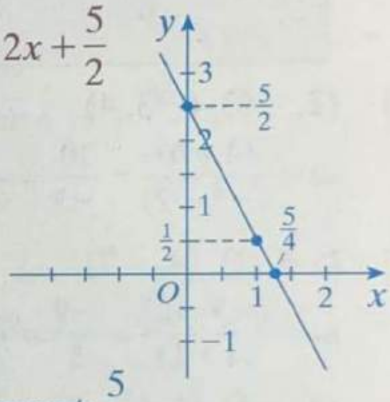
(4) $y=-x+\frac{1}{2}$



y-intercept: $\frac{1}{2}$

slope: -1

(6) $y=-2x+\frac{5}{2}$

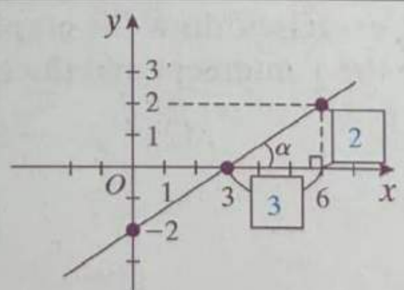


y-intercept: $\frac{5}{2}$

slope: -2

Let's think about this!

$y=\frac{2}{3}x-2$



The angle α (where $0° \leq \alpha \leq 180°$) that a line makes with the positive direction of the x-axis is called the *inclination* of the line. The tangent of this angle, $\tan\alpha$, is the slope of the line. Therefore, from the graph,

$$m = \tan\alpha = \frac{2}{3}$$

Answers: 2, 3, $\frac{2}{3}$

KUMON

M 152 a

Equations of Straight Lines 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

The equation of a straight line with slope m and a y-intercept b is expressed as:

Slope Intercept Form $\boxed{y = mx + b}$

Obtain the equation of each of the following straight lines.

(1) A line passing through point $(0, 3)$ with slope -2.

[Sol] Since the line passes through point $(0, 3)$, the y-intercept is 3.
Therefore, $y = -2x + 3$

(2) A line passing through point $(3, 0)$ with slope -2.

[Sol] Letting $y = -2x + b$ be the equation of the line,
Since it passes through point $(3, 0)$, substituting into the equation,
$0 = -2 \cdot 3 + b$
$b = 6$
Therefore, $y = -2x + 6$

(3) A line passing through point $(2, 3)$ with slope 2.

[Sol] Letting $y = 2x + b$ be the equation of the line,
Since it passes through point $(2, 3)$, substituting into the equation,
$3 = 2 \cdot 2 + b$
$b = -1$
Therefore, $y = 2x - 1$

(4) A line passing through point $(-1, 2)$ with slope $\frac{1}{2}$.

[Sol] Letting $y = \frac{1}{2}x + b$ be the equation of the line,

Since it passes through point $(-1, 2)$, substituting into the equation,

$2 = \frac{1}{2} \cdot (-1) + b$

$b = \frac{5}{2}$

Therefore, $y = \frac{1}{2}x + \frac{5}{2}$

(5) A line passing through point $(-2, -3)$ with slope $-\frac{1}{4}$.

[Sol] Letting $y = -\frac{1}{4}x + b$ be the equation of the line,

Since it passes through point $(-2, -3)$, substituting into the equation,

$-3 = -\frac{1}{4} \cdot (-2) + b$

$b = -\frac{7}{2}$

Therefore, $y = -\frac{1}{4}x - \frac{7}{2}$

(6) A line passing through point (x_1, y_1) with slope m.

[Sol] Let the equation of the line be: $y = mx + b$ …①

Since ① passes through point (x_1, y_1),

$y_1 = mx_1 + b$ …②

From ①−②, $y - y_1 = \boxed{m(x - x_1)}$

M 153 a Equations of Straight Lines 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

Point-Slope Formula

The equation of a straight line passing through a point (x_1, y_1) with slope m is expressed as:

$$y - y_1 = m(x - x_1)$$

1. Obtain the equation of each of the following straight lines.

(1) A line passing through point $(1, 2)$, with slope 2.

[Sol] $y - 2 = 2(x - 1)$

$y = 2x$

(2) A line passing through point $(-4, 1)$, with slope 3.

[Sol] $y - 1 = 3(x + 4)$

$y = 3x + 13$

(3) A line passing through point $(4, -1)$, with slope -3.

[Sol] $y + 1 = -3(x - 4)$

$y = -3x + 11$

(4) A line passing through point $(-4, -1)$, with slope $\frac{1}{2}$.

[Sol] $y + 1 = \frac{1}{2}(x + 4)$

$y = \frac{1}{2}x + 1$

(5) A line passing through point $(1, 2)$, with slope 0.

[Sol] $y - 2 = 0$

$y = 2$

M 153 b

2. In each of the following exercises, find the equation of the line that passes through the two given points.

Ex. $(-3, 1), (3, 4)$

[Sol] $m=\dfrac{4-1}{3+3}=\dfrac{3}{6}=\dfrac{1}{2}$

Substituting this value, and the coordinates of either one of the given points into the Point-Slope Formula.

$$y-1=\frac{1}{2}(x+3)$$ ← Substituted point $(-3, 1)$

$$y=\frac{1}{2}x+\frac{5}{2}$$

(1) $(-2, 4), (1, 7)$

[Sol] $m=\dfrac{7-4}{1+2}=\dfrac{3}{3}=1$

$$y-4=1(x+2)$$ ← Substituted point $(-2, 4)$

$$y=x+6$$

(2) $(-1, -2), (3, 4)$

[Sol] $m=\dfrac{4+2}{3+1}=\dfrac{6}{4}=\dfrac{3}{2}$

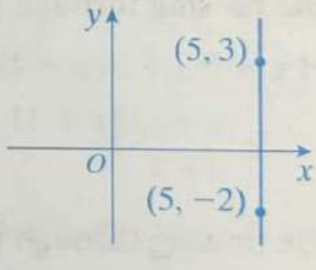
$$y-4=\frac{3}{2}(x-3)$$ ← Substituted point $(3, 4)$

$$y=\frac{3}{2}x-\frac{1}{2}$$

(3) $(5, 3), (5, -2)$

[Sol] The slope, m is undefined.

Therefore, as shown on the graph,



The equation of the line is $x=5$.

(4) $(x_1, y_1), (x_2, y_2)$

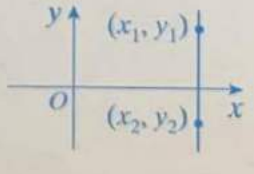
a) When $x_1 \neq x_2$

[Sol] $m=\dfrac{y_2-y_1}{x_2-x_1}$, $y-y_1=\dfrac{y_2-y_1}{x_2-x_1}(x-x_1)$ ← Substituted point (x_1, y_1)

$$\left[y=\frac{y_2-y_1}{x_2-x_1}x+\frac{x_2y_1-x_1y_2}{x_2-x_1}\right]$$

b) When $x_1 = x_2$

[Sol] From the graph on the right,



$$x=x_1$$

$$[x=x_2]$$

M 154 a Equations of Straight Lines 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

1. Using the Point-Slope Formula, obtain the value of a at which the three given points all lie on the same line.

(1) $A(3, 1), B(-2, 4), C(a, -2)$

[Sol] Point C must be on the line formed by points A and B.

The slope of AB is $\boxed{-\frac{3}{5}}$.

Using the coordinates of point A, and the slope,

The equation of line AB is: $y-\boxed{1}=\boxed{-\frac{3}{5}}(x-\boxed{3})$

$$y=\boxed{-\frac{3}{5}x+\frac{14}{5}} \quad \ldots ①$$

Since $C(a,-2)$ must be on line ①,

$$\boxed{-2}=\boxed{-\frac{3}{5}}a+\boxed{\frac{14}{5}}$$

$$a=\boxed{8}$$

(2) $A(-3, 7), B(a, -2), C(5, -4)$

[Sol] Point B must be on the line formed by points A and C.

The slope of AC is $-\frac{11}{8}$.

Using the coordinates of point A, and the slope,

The equation of line AC is: $y-7=-\frac{11}{8}(x+3)$

$$y=-\frac{11}{8}x+\frac{23}{8} \quad \ldots ①$$

Since $B(a, -2)$ must be on line ①,

$$-2=-\frac{11}{8}a+\frac{23}{8}$$

$$a=\frac{39}{11}$$

2. Obtain the equation of a line that passes through points $A(a, 0)$ and $B(0, b)$.

[Sol] The slope of AB is: $m = \frac{b-0}{0-a} = -\frac{b}{a}$

The equation of the line can be written as:

$$y = -\frac{b}{a}(x-a) = -\frac{b}{a}x + b$$

Rearranging the answer of the above exercise,

Intercept Form $\boxed{\frac{x}{a} + \frac{y}{b} = 1}$

This form is used to express the equation of a line that crosses the x-axis at $A(a, 0)$ and the y-axis at $B(0, b)$, where $ab \neq 0$.

3. In each of the following equations, use the above formula to obtain the equation of the line passing through the two given points.

(1) $(2, 0)$ and $(0, -3)$

[Sol] $\frac{x}{2} + \frac{y}{(-3)} = 1$

$\frac{x}{2} - \frac{y}{3} = 1$

$[y = \frac{3}{2}x - 3$ or $y = \frac{3}{2}(x-2)]$

(2) $(3, 0)$ and $(0, 1)$

[Sol] $\frac{x}{3} + \frac{y}{1} = 1$

$\frac{x}{3} + y = 1$

$[y = -\frac{1}{3}x + 1$ or $y = -\frac{1}{3}(x-3)]$

(3) $(-1, 0)$ and $(0, -4)$

[Sol] $\frac{x}{(-1)} + \frac{y}{(-4)} = 1$

$-\frac{x}{1} - \frac{y}{4} = 1$

$-x - \frac{y}{4} = 1$

$[y = -4x - 4$ or $y = -4(x+1)]$

(4) $(-5, 0)$ and $(0, 6)$

[Sol] $\frac{x}{(-5)} + \frac{y}{6} = 1$

$-\frac{x}{5} + \frac{y}{6} = 1$

$[y = \frac{6}{5}x + 6$ or $y = \frac{6}{5}(x+5)]$

M 155 a Equations of Straight Lines 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

From M151b, given the angle α (where $0° \le \alpha < 180°$) that a line makes with the positive direction of the x-axis, the slope of the line can be expressed as:

$$m = \boxed{\tan}\,\alpha$$

Answer: tan

1. In each of the following exercises, obtain the angle, α, which the given line forms with the positive direction of the x-axis.

(1) $y = \sqrt{3}x + 2$

[Sol] $\tan\alpha = \sqrt{3}$ $\quad \therefore \alpha = 60°$

(2) $y = -x + 3$

[Sol] $\tan\alpha = -1$ $\quad \therefore \alpha = 135°$

(3) $y = \dfrac{\sqrt{3}}{3}x + 1$

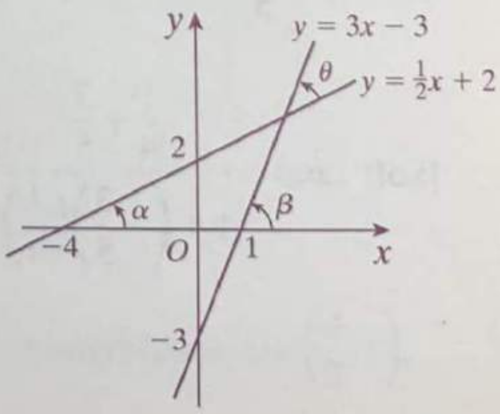
[Sol] $\tan\alpha = \dfrac{\sqrt{3}}{3}$ $\quad \therefore \alpha = 30°$

2. Obtain the angle θ formed from the line $y = \dfrac{1}{2}x + 2$ to the line $y = 3x - 3$.

[Sol] Letting α be the angle which line $y = \dfrac{1}{2}x + 2$ forms with the x-axis, and

Letting β be the angle which line $y = 3x - 3$ forms with the x-axis,

$\tan\alpha = \boxed{\dfrac{1}{2}}$, $\quad \tan\beta = \boxed{3}$ $\quad \ldots ①$



From the graph,

$\theta = \beta - \boxed{\alpha}$

From the Addition Theorem,

$$\tan\theta = \tan(\beta - \boxed{\alpha}) = \boxed{\dfrac{\tan\beta - \tan\alpha}{1 + \tan\alpha\tan\beta}}$$

Substituting ① into the above equation, $\tan\theta = \boxed{1}$. Therefore, $\theta = \boxed{45}°$.

M 155 b

Given that θ is the angle formed from the line $y = mx + b$ to line $y = m'x + b'$, then

$$\boxed{\tan\theta = \frac{m' - m}{1 + mm'}}$$

3. Find the angle formed between each of the following pairs of lines.

(1) $x + 2y + 3 = 0, \quad x - 3y + 1 = 0$

[Sol] Using the above formula,

$$\tan\theta = \frac{\frac{1}{3} - \left(-\frac{1}{2}\right)}{1 + \left(-\frac{1}{2}\right) \cdot \frac{1}{3}} = 1 \qquad \theta = 45° \ [135°]$$

(2) $2\sqrt{3}x - y = 0, \quad 7x + \sqrt{3}y - 5 = 0$

[Sol] $$\tan\theta = \frac{-\frac{7}{\sqrt{3}} - 2\sqrt{3}}{1 + 2\sqrt{3}\left(-\frac{7}{\sqrt{3}}\right)} = \frac{\sqrt{3}}{3} \qquad \theta = 30° \ [150°]$$

(3) $3x + 5y + \frac{7}{5} = 0, \quad x - 4y + 14 = 0$

[Sol] $$\tan\theta = \frac{\frac{1}{4} + \frac{3}{5}}{1 + \left(-\frac{3}{5}\right)\left(\frac{1}{4}\right)} = 1 \qquad \theta = 45° \ [135°]$$

M 156 a Equations of Straight Lines 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. Find the coordinates of the point of intersection of each of the following pairs of lines.

(1) $\begin{cases} y=2x-5 & \dots ① \\ y=-\frac{1}{3}x+2 & \dots ② \end{cases}$

From ① and ②,

$$2x-5=-\frac{1}{3}x+2$$

$$x=3$$

Substituting this into ①,

$$y=1$$

Therefore, the coordinates of the point of intersection are (3, 1).

(2) $\begin{cases} 2x-3y=5 & \dots ① \\ x-5y+1=0 & \dots ② \end{cases}$

From ① − ② × 2,

$$y=1$$

Substituting this into ②,

$$x=4$$

Therefore, the coordinates of the point of intersection are (4, 1).

(3) $\begin{cases} 3x+4y=8 & \dots ① \\ 6x-5y=3 & \dots ② \end{cases}$

From ① × 2 − ②,

$$y=1$$

Substituting this into ①,

$$x=\frac{4}{3}$$

Therefore, the coordinates of the point of intersection are $\left(\frac{4}{3}, 1\right)$.

2. Find the coordinates of the point of intersection P of the following pair of lines. Then, find the equation of a line that passes through point P and through point $(2, -2)$.

$2x-3y-1=0 \quad \cdots ①$
$4x-y+3=0 \quad \cdots ②$

[Sol] From ① + $(-3)\times$ ②,
$-10x-10=0$
$x=-1$
Substituting this value into ①,
$y=-1$
Therefore, the coordinates of P are $(-1, -1)$.

The slope of the line formed by $P(-1, -1)$ and $(2, -2)$ is $m=\dfrac{-2+1}{2+1}=-\dfrac{1}{3}$

Using the slope and P, $\quad y+1=-\dfrac{1}{3}(x+1)$

$\therefore\ y=-\dfrac{1}{3}x-\dfrac{4}{3}$

3. Find the coordinates of the point of intersection P of the following pair of lines. Then, find the equation of a line that passes through point P and through point $(-4, 3)$.

$6x-y+5=0 \quad \cdots ①$
$2x+3y-7=0 \quad \cdots ②$

[Sol] From $3\times$ ① + ②,
$20x+8=0$
$x=-\dfrac{2}{5}$
Substituting this value into ①,
$y=\dfrac{13}{5}$
Therefore, the coordinates of P are $\left(-\dfrac{2}{5}, \dfrac{13}{5}\right)$.

The slope of the line formed by $P\left(-\dfrac{2}{5}, \dfrac{13}{5}\right)$ and $(-4, 3)$ is $m=\dfrac{3-\dfrac{13}{5}}{-4+\dfrac{2}{5}}=-\dfrac{1}{9}$

Using the slope and $(-4, 3)$, $\quad y-3=-\dfrac{1}{9}(x+4)$

$\therefore\ y=-\dfrac{1}{9}x+\dfrac{23}{9}$

M 157 a Equations of Straight Lines 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

If two lines with slopes m_1 and m_2 are parallel, then their slopes are equal, i.e.:

$$m_1 = \boxed{m_2}$$

(Conversely, if two lines have equal slopes, then the lines are parallel.)

Answer: m_2

1. In each of the following exercises, find the equation of a line which is parallel to the given line and which passes through point $(4, -3)$.

(1) $y = -\sqrt{3}x + 6$

[Sol] The slope of the given line is $-\sqrt{3}$.

Therefore, since the slope of the parallel line is also $-\sqrt{3}$,

$$y + 3 = -\sqrt{3}(x - 4)$$

$$\therefore\ y = -\sqrt{3}x + 4\sqrt{3} - 3$$

(2) $3x - 5y - 1 = 0$

[Sol] The slope of the given line is $\frac{3}{5}$.

Therefore, since the slope of the parallel line is also $\frac{3}{5}$,

$$y + 3 = \frac{3}{5}(x - 4) \qquad \therefore\ y = \frac{3}{5}x - \frac{27}{5} \quad [3x - 5y - 27 = 0]$$

(3) $3x + 2y + 5 = 0$

[Sol] The slope of the given line is $-\frac{3}{2}$.

Therefore, since the slope of the parallel line is also $-\frac{3}{2}$,

$$y + 3 = -\frac{3}{2}(x - 4) \qquad \therefore\ y = -\frac{3}{2}x + 3 \quad [3x + 2y - 6 = 0]$$

(4) $x = 0$

[Sol] Line $x = 0$ coincides with the y-axis.

$$\therefore\ x = 4$$

(5) $y = -2$

[Sol] Line $y = -2$ is parallel to the x-axis.

$$\therefore\ y = -3$$

2. Given the line $l_1 : 5y - 3kx + 6 = 0$, determine the value of k at which l_1 is parallel to $l_2 : 2y + 4x - 1 = 0$.

[Sol] The slope of l_1 is $m_1 = \boxed{\frac{3}{5}k}$. The slope of l_2 is $m_2 = \boxed{-2}$.

l_1 would be parallel to l_2, if and only if $m_1 = \boxed{m_2}$.

$$\boxed{\frac{3}{5}k} = \boxed{-2}$$

$$k = \boxed{-\frac{10}{3}}$$

3. Given the line $l_1 : 2y - \frac{k}{2}x - 3 = 0$, determine the value of k at which l_1 is parallel to $l_2 : 3y - 5x + 2 = 0$.

[Sol] The slope of l_1 is $m_1 = \frac{k}{4}$. The slope of l_2 is $m_2 = \frac{5}{3}$.

l_1 would be parallel to l_2, if and only if $m_1 = m_2$.

$$\frac{k}{4} = \frac{5}{3}$$

$$k = \frac{20}{3}$$

4. Given the line $l_1 : 4y + 5k^2x - 3 = 0$, determine the value of k at which l_1 is parallel to $l_2 : 2y - 3kx + 4 = 0$.

[Sol] The slope of l_1 is $m_1 = -\frac{5}{4}k^2$. The slope of l_2 is $m_2 = \frac{3}{2}k$.

l_1 would be parallel to l_2, if and only if $m_1 = m_2$.

$$-\frac{5}{4}k^2 = \frac{3}{2}k$$

$$-\frac{5}{4}k^2 - \frac{3}{2}k = 0$$

$$5k^2 + 6k = 0$$

$$k(5k + 6) = 0$$

$$k = 0, -\frac{6}{5}$$

5. Obtain the value of a at which the three given points all lie on the same line.

$$A(2, 1), B(-3, 4), C(a, -5)$$

[Sol] In order for the three points to lie on the same line, the slope of AB must equal the slope of BC.

The slope of AB is $\boxed{-\frac{3}{5}}$ . . . ① The slope of BC is $\boxed{-\frac{9}{a+3}}$ . . . ②

Since ① and ② must be equal, $\boxed{-\frac{3}{5}} = \boxed{-\frac{9}{a+3}}$

$$3a + 9 = 45$$

$$a = \boxed{12}$$

Equations of Straight Lines 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

In order for two lines to intersect at right angles, (to be *perpendicular*) their slopes, m_1 and m_2, must be negative reciprocals of each other:

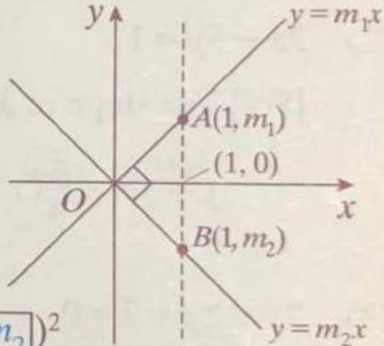
$$m_2 = -\frac{1}{m_1}$$

This means that the product of their slopes equals -1:

$$m_1 m_2 = -1$$

1. Prove that if line $y = m_1 x$ is perpendicular to line $y = m_2 x$, then $m_1 m_2 = -1$.

[Sol] A line drawn parallel to the y-axis passing through point $(1, 0)$ intersects lines $y = m_1 x$ and $y = m_2 x$ at the following points:



$$A(1, m_1)$$
$$B(1, m_2)$$

Since $AO \perp BO$, $OA^2 + \boxed{OB^2} = AB^2$

Therefore, $(1^2 + m_1{}^2) + (1^2 + \boxed{m_2{}^2}) = (m_1 - \boxed{m_2})^2$

$$2 + m_1{}^2 + m_2{}^2 = \boxed{m_1{}^2 - 2m_1m_2 + m_2{}^2}$$

$$\therefore m_1 m_2 = \boxed{-1}$$

2. Prove that if $m_1 m_2 = -1$, then line $y = m_1 x$ is perpendicular to line $y = m_2 x$.

[Sol] Since $m_1 m_2 = -1$,

$AB^2 = (m_1 - m_2)^2 = \boxed{m_1{}^2 - 2m_1m_2 + m_2{}^2 = m_1{}^2 + m_2{}^2 + 2}$

$OA^2 + OB^2 = (1^2 + m_1{}^2) + (1^2 + \boxed{m_2{}^2}) = m_1{}^2 + m_2{}^2 + 2$

Therefore, $OA^2 + \boxed{OB^2} = \boxed{AB^2}$

Since $\triangle OAB$ is a right triangle, OA is perpendicular to $\boxed{OB}$ (i.e. $OA \perp \boxed{OB}$).

Thus, line $y = m_1 x$ is perpendicular to line $y = m_2 x$.

> If two lines with slopes m_1 and m_2 are perpendicular, then the product of their slopes is -1.
>
> $\boxed{m_1 m_2 = -1}$

3. For each of the following lines, find the equation of the line which is perpendicular to it and which passes through point $(4, -3)$.

(1) $y = -\sqrt{3}x + 6$

[Sol] To find the slope m of the perpendicular line,

Solving $\boxed{-\sqrt{3}}\, m = -1$,

$$m = \boxed{\frac{\sqrt{3}}{3}}$$

Therefore, $y + 3 = \frac{\sqrt{3}}{3}(x-4)$ $\therefore\ y = \frac{\sqrt{3}}{3}x - \frac{4\sqrt{3}}{3} - 3$

(2) $3x - 5y = 1$

[Sol] The slope of $3x - 5y = 1$ is $\frac{3}{5}$.

$y + 3 = -\frac{5}{3}(x-4)$

$\therefore\ y = -\frac{5}{3}x + \frac{11}{3}$ $[5x + 3y - 11 = 0]$

(3) $3x + 2y + 5 = 0$

[Sol] The slope of $3x + 2y + 5 = 0$ is $-\frac{3}{2}$.

$y + 3 = \frac{2}{3}(x-4)$

$\therefore\ y = \frac{2}{3}x - \frac{17}{3}$ $[2x - 3y - 17 = 0]$

(4) $x = 0$

[Sol] A line that is perpendicular to $x = 0$ is parallel to the x-axis.

$\therefore\ y = -3$

(5) $y = -2$

[Sol] A line that is perpendicular to $y = -2$ is parallel to the y-axis.

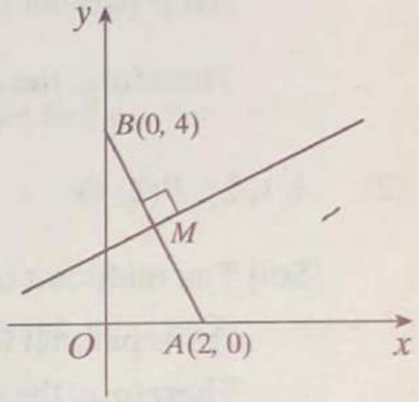
$\therefore\ x = 4$

M 159 a Equations of Straight Lines 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. Obtain the equation of the line that is perpendicular to and *bisecting* (i.e. dividing into two equal parts) line segment AB with points $A(2, 0)$ and $B(0, 4)$.



[Sol] Letting M be the midpoint of AB, we need to find the equation of the line that passes through M and is perpendicular to AB.

The coordinates of M are $(\boxed{1}, \boxed{2})$.

Therefore, the equation of the line we are looking for can be expressed as:

$$y - 2 = m(\boxed{x - 1}) \quad \cdots ①$$

Since the slope of AB is $\boxed{-2}$, calculating the slope of the perpendicular line,

$$m \times \boxed{(-2)} = -1$$

$$m = \boxed{\frac{1}{2}}$$

Substituting this into ①,

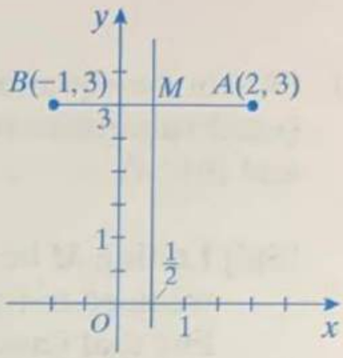
$$y - 2 = \boxed{\frac{1}{2}(x - 1)}$$

$$\therefore\ y = \boxed{\frac{1}{2}x + \frac{3}{2}}$$

M 159 b

2. Obtain the equation of a line that is perpendicular to and bisecting line segment AB connecting each of the following pairs of points.

(1) $A(2, 3)$, $B(-1, 3)$

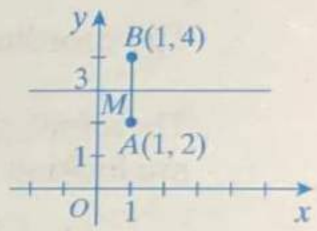


[Sol] The midpoint of AB is $M\left(\frac{1}{2}, 3\right)$.

AB is parallel to the x-axis.

Therefore, the equation of the line is $x = \frac{1}{2}$.

(2) $A(1, 2)$, $B(1, 4)$

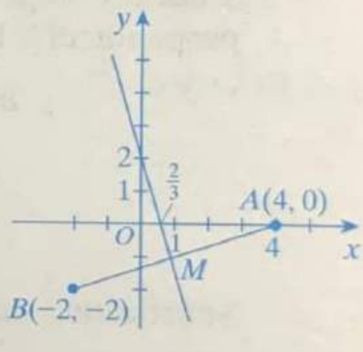


[Sol] The midpoint of AB is $M(1, 3)$.

AB is parallel to the y-axis.

Therefore, the equation of the line is $y = 3$.

(3) $A(4, 0)$, $B(-2, -2)$



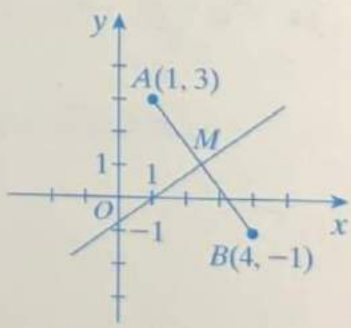
[Sol] The midpoint of AB is $M(1, -1)$.

The slope of AB is $\frac{1}{3}$.

Therefore, the equation of the line is:

$$y + 1 = -3(x - 1)$$
$$y = -3x + 2$$

(4) $A(1, 3)$, $B(4, -1)$



[Sol] The midpoint of AB is $M\left(\frac{5}{2}, 1\right)$.

The slope of AB is $-\frac{4}{3}$.

Therefore, the equation of the line is: $y - 1 = \frac{3}{4}\left(x - \frac{5}{2}\right)$

$$y = \frac{3}{4}x - \frac{7}{8}$$

Equations of Straight Lines 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

1. Obtain the equation of each of the following lines.

(1) A line parallel to $3x+y-7=0$ with y-intercept 5.

[Sol] The slope of $3x+y-7=0$ is -3.

$\therefore\ y=-3x+5\quad [3x+y-5=0]$

(2) A line passing through point $(2, 3)$ and parallel to $3x-y=7$.

[Sol] The slope of $3x-y=7$ is 3.

$y-3=3(x-2)\qquad \therefore\ y=3x-3\quad [3x-y-3=0]$

(3) A line passing through point $(-3, 5)$ and perpendicular to $2x+4y=1$.

[Sol] The slope of $2x+4y=1$ is $-\frac{1}{2}$.

$y-5=2(x+3)\qquad \therefore\ y=2x+11\quad [2x-y+11=0]$

(4) A line perpendicular to $2x-y=3$ with y-intercept 3.

[Sol] The slope of $2x-y=3$ is 2.

$\therefore\ y=-\frac{1}{2}x+3\quad [x+2y-6=0]$

(5) A line passing through point $(-3, -1)$ and perpendicular to $2x-3y+2=0$.

[Sol] The slope of $2x-3y+2=0$ is $\frac{2}{3}$.

$y+1=-\frac{3}{2}(x+3)\qquad \therefore\ y=-\frac{3}{2}x-\frac{11}{2}\quad [3x+2y+11=0]$

(6) A line passing through point $(2, 5)$ and perpendicular to the line passing through the two points $(1, 3)$ and $(3, -1)$.

[Sol] The slope of the line passing through $(1, 3)$ and $(3, -1)$ is

$m_1=\frac{-1-3}{3-1}=-2.$

The slope of the line perpendicular to the line passing through $(1, 3)$ and $(3, -1)$ is $m_2=\frac{1}{2}$.

Using point $(2, 5)$ and slope m_2, $y-5=\frac{1}{2}(x-2)\qquad \therefore\ y=\frac{1}{2}x+4$

2. Find the value of k at which the following two lines are parallel.

$$3y + 2k^2x - 3 = 0 \quad \ldots ①$$
$$6y - 7kx + 1 = 0 \quad \ldots ②$$

[Sol] The slope of line ① is $m_1 = -\frac{2}{3}k^2$. The slope of line ② is $m_2 = \frac{7}{6}k$.

① and ② are parallel when $m_1 = m_2$.

$$-\frac{2}{3}k^2 = \frac{7}{6}k$$
$$-\frac{2}{3}k^2 - \frac{7}{6}k = 0$$
$$4k^2 + 7k = 0$$
$$k(4k + 7) = 0$$
$$k = 0, -\frac{7}{4}$$

3. Obtain the value of a at which the three given points all lie on the same line.

$$A(5, 4), B(-4, 1), C(a, -3)$$

[Sol] In order for the three points to lie on the same line, the slope of AB must equal the slope of BC.

The slope of AB is $\frac{1}{3}$ $\quad \ldots ①$ The slope of BC is $-\frac{4}{a+4}$ $\quad \ldots ②$

Since ① and ② must be equal, $\frac{1}{3} = -\frac{4}{a+4}$

$$a + 4 = -12$$
$$a = -16$$

M 161 a — Equations of Straight Lines 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

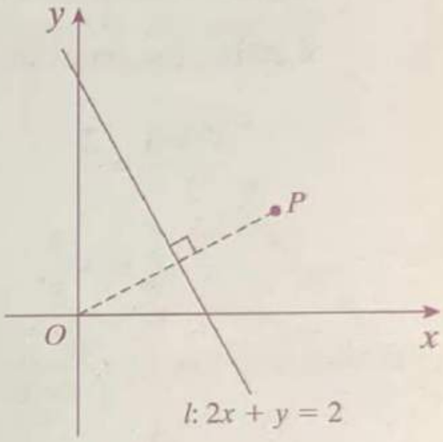
1. Obtain the coordinates of point P which is symmetric to the origin O with respect to line l whose equation is $2x+y=2$, by using the following two methods.

(1) The equation of line l can be written as:

$y=\boxed{-2x+2}$ $\quad\ldots①$

Since OP and l are perpendicular, the equation of OP is:

$y=\boxed{\frac{1}{2}x}$ $\quad\ldots②$



Solving ① and ② as simultaneous equations, we can obtain the x and y coordinates of the point of intersection,

$\left(\boxed{\frac{4}{5}}, \boxed{\frac{2}{5}}\right)$

Therefore, the coordinates of P are $\left(\boxed{\frac{8}{5}}, \boxed{\frac{4}{5}}\right)$.

(2) Let the coordinates of P be (X, Y).
Since OP and l are perpendicular,

$\frac{Y}{\boxed{X}}\cdot\boxed{(-2)}=-1$

$\therefore\ Y=\boxed{\frac{1}{2}X}$ $\quad\ldots①$

Since the midpoint $\left(\frac{X}{\boxed{2}}, \frac{Y}{\boxed{2}}\right)$ of OP lies on line l,

$2\cdot\frac{X}{\boxed{2}}+\frac{Y}{\boxed{2}}=2$ $\quad\ldots②$

From ① and ②,

$X=\boxed{\frac{8}{5}},\quad Y=\boxed{\frac{4}{5}}$

Therefore, the coordinates of P are $\left(\boxed{\frac{8}{5}}, \boxed{\frac{4}{5}}\right)$.

2. Obtain the coordinates of point B which is symmetric to point $A(3, 2)$ with respect to the line $2x + y = 2$.

[Sol] The equation of the line perpendicular to $2x + y = 2$ $\quad \ldots ①$

which passes through point $A(3, 2)$ is $y - 2 = \frac{1}{2}(x - 3)$ $\quad \ldots ②$

The coordinates of the point of intersection of ① and ② are $\left(\frac{3}{5}, \frac{4}{5}\right)$.

Letting the coordinates of point B be (X, Y),

$$\frac{X + 3}{2} = \frac{3}{5}, \qquad \frac{Y + 2}{2} = \frac{4}{5}$$

$$\therefore X = -\frac{9}{5}, \qquad Y = -\frac{2}{5}$$

Ans. $B\left(-\frac{9}{5}, -\frac{2}{5}\right)$

3. Obtain the coordinates of point B which is symmetric to point $A(0, 4)$ with respect to line $2x - y - 1 = 0$.

[Sol] The equation of the line perpendicular to $2x - y - 1 = 0$ $\quad \ldots ①$

which passes through point $A(0, 4)$ is $y - 4 = -\frac{1}{2}x$ $\quad \ldots ②$

The coordinates of the point of intersection of ① and ② are $(2, 3)$.

Letting the coordinates of point B be (X, Y),

$$\frac{X + 0}{2} = 2, \quad \frac{Y + 4}{2} = 3$$

$$\therefore X = 4, \qquad Y = 2$$

Ans. $B(4, 2)$

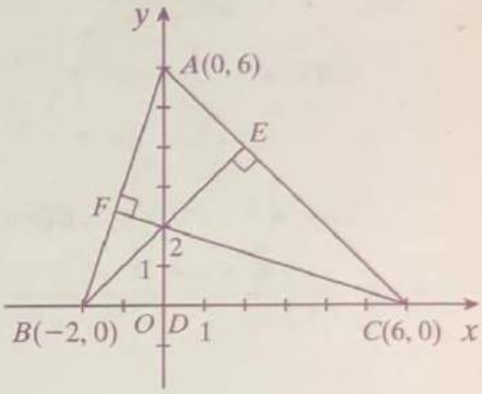
M 162 a Equations of Straight Lines 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

1. Given $\triangle ABC$ with vertices $A(0, 6)$, $B(-2, 0)$ and $C(6, 0)$, and whose perpendicular lines dropped from A, B and C are AD, BE and CF, respectively, solve the following exercises.

(1) Obtain the equations of the perpendicular lines AD, BE and CF.



[Sol] AD: The equation of AD is $x = 0$.

BE: BE passes through point $B(-2, 0)$ and is perpendicular to AC.
Since AC has slope -1,
BE has slope $\boxed{1}$.
Therefore, the equation of BE is $y = \boxed{x + 2}$.

CF: CF passes through point $C(6, 0)$ and is perpendicular to AB.
Since AB has slope 3, CF has slope $\boxed{-\frac{1}{3}}$.

Therefore, the equation of CF is $y = \boxed{-\frac{1}{3}(x-6)}$.

Ans. AD: $x = 0$, BE: $y = \boxed{x + 2}$, CF: $y = \boxed{-\frac{1}{3}x + 2}$

(2) Obtain the coordinates of the point of intersection of AD and BE.

[Sol] AD: $x = 0$ …①
BE: $y = x + 2$ …②
From ① and ②, $x = 0$ and $y = 2$.
Therefore, the point of intersection is $(0, 2)$.

(3) Obtain the coordinates of the point of intersection of BE and CF.

[Sol] BE: $y = x + 2$ …①
CF: $y = -\frac{1}{3}x + 2$ …②
From ① and ②, $x = 0$ and $y = 2$.
Therefore, the point of intersection is $(0, 2)$.

(4) Obtain the coordinates of the point of intersection of CF and AD.

[Sol] CF: $y = -\frac{1}{3}x + 2$ …①
AD: $x = 0$ …②
From ① and ②, $x = 0$ and $y = 2$.
Therefore, the point of intersection is $(0, 2)$.

2. Given $\triangle ABC$ with vertices $A(3, 4)$, $B(0, -8)$ and $C(8, 0)$, and whose perpendicular lines dropped from A, B and C are AP, BQ and CR, respectively, solve the following exercises.

(1) Obtain the equations of the perpendicular lines AP, BQ and CR.

AP: $y - 4 = -(x - 3)$

$\therefore y = -x + 7$

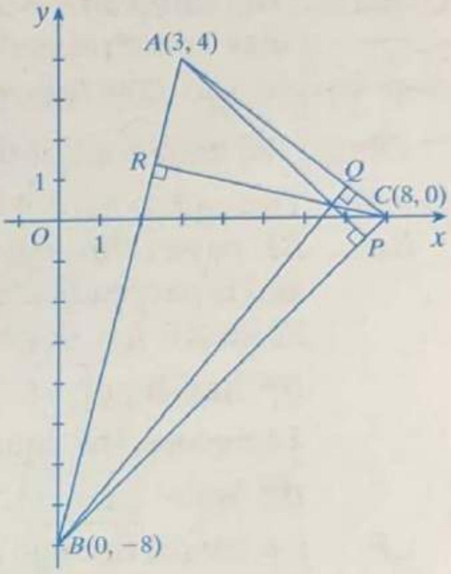
BQ: $y + 8 = \frac{5}{4}x$

$\therefore y = \frac{5}{4}x - 8$

CR: $y = -\frac{1}{4}(x - 8)$

$\therefore y = -\frac{1}{4}x + 2$

Ans. AP: $y = -x + 7$, BQ: $y = \frac{5}{4}x - 8$,
CR: $y = -\frac{1}{4}x + 2$



(2) Obtain the coordinates of the point of intersection of AP and BQ.

[Sol] AP: $y = -x + 7$ …①

BQ: $y = \frac{5}{4}x - 8$ …②

From ① and ②, $x = \frac{20}{3}$ and $y = \frac{1}{3}$.

Therefore, the point of intersection is $\left(\frac{20}{3}, \frac{1}{3}\right)$.

(3) Obtain the coordinates of the point of intersection of BQ and CR.

[Sol] BQ: $y = \frac{5}{4}x - 8$ …①

CR: $y = -\frac{1}{4}x + 2$ …②

From ① and ②, $x = \frac{20}{3}$ and $y = \frac{1}{3}$.

Therefore, the point of intersection is $\left(\frac{20}{3}, \frac{1}{3}\right)$.

(4) Obtain the coordinates of the point of intersection of CR and AP.

[Sol] CR: $y = -\frac{1}{4}x + 2$ …①

AP: $y = -x + 7$ …②

From ① and ②, $x = \frac{20}{3}$ and $y = \frac{1}{3}$.

Therefore, the point of intersection is $\left(\frac{20}{3}, \frac{1}{3}\right)$.

KUMON

M 163 a Equations of Straight Lines 2

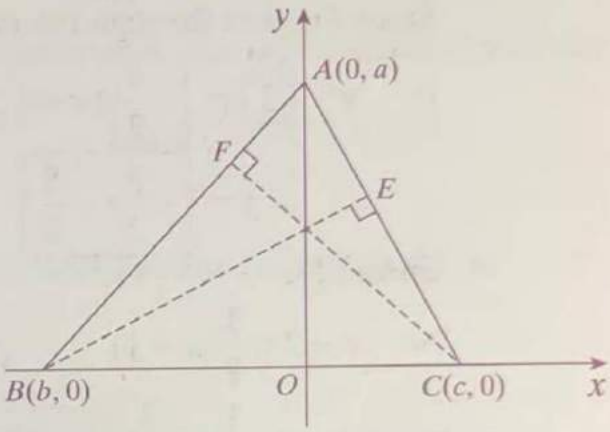
Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. Prove that all of the perpendicular lines dropped from the vertices to the opposite sides of $\triangle ABC$ intersect at one point.

When $\triangle ABC$ is a right triangle, all three perpendicular lines intersect at the vertex of the right angle.

In all other cases, place side BC of $\triangle ABC$ on the x-axis, place the perpendicular line from A to BC on the y-axis, and let A, B and C be at $A(0, a)$, $B(b, 0)$ and $C(c, 0)$ where $abc \neq 0$.



Since the slope of AC is $\boxed{-\frac{a}{c}}$, the slope of the perpendicular line BE, from B to AC is $\boxed{\frac{c}{a}}$.

Therefore, the equation of perpendicular line BE is: $\boxed{y=\frac{c}{a}x-\frac{bc}{a}}$ . . . ①

Similarly, the equation of the perpendicular line CF from C to AB is:

$\boxed{y=\frac{b}{a}x-\frac{bc}{a}}$ . . . ②

From ① and ②, the x-coordinate of the point of intersection of lines ① and ② is $x = \boxed{0}$.

BE and CF intersect on the y-axis, i.e. on AO.

Therefore, the three perpendicular lines, AO, BE and CF, all intersect at one point.

Terminology: The point at which all three perpendicular lines of a triangle intersect is called the **orthocenter** of the triangle.

2. Given $\triangle ABC$ with vertices $A(0, 4)$, $B(-6, 0)$ and $C(2, 0)$, where l, m and n are the perpendicular lines bisecting AB, AC and BC, respectively, solve the following exercises.

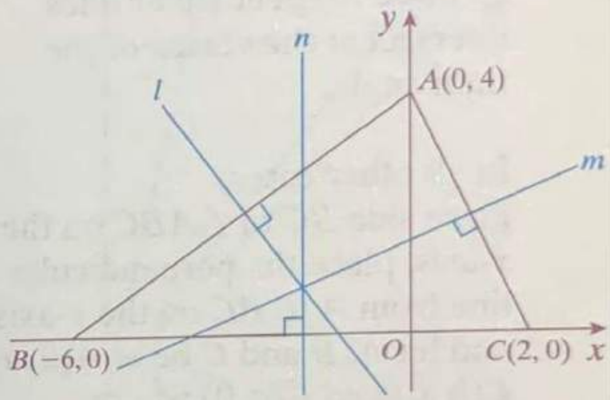
(1) Obtain the equations of l, m and n.

[Sol] The slope of AB is $\boxed{\frac{2}{3}}$. Therefore, the slope of l is $\boxed{-\frac{3}{2}}$.

Since l passes through the midpoint $(\boxed{-3}, \boxed{2})$ of AB,

$$l: \quad y - \boxed{2} = \boxed{-\frac{3}{2}}(x + \boxed{3})$$

$$\therefore y = \boxed{-\frac{3}{2}x - \frac{5}{2}}$$



Similarly,

$$m: \quad y - 2 = \frac{1}{2}(x - 1)$$

$$\therefore y = \frac{1}{2}x + \frac{3}{2}$$

$$n: \quad x = -2$$

Ans. l: $\boxed{y = -\frac{3}{2}x - \frac{5}{2}}$, m: $\boxed{y = \frac{1}{2}x + \frac{3}{2}}$, n: $\boxed{x = -2}$

(2) Obtain the coordinates of the point of intersection of lines l and m.

[Sol] l: $y = -\frac{3}{2}x - \frac{5}{2}$ …①, m: $y = \frac{1}{2}x + \frac{3}{2}$ …②

From ① and ②, $x = -2$ and $y = \frac{1}{2}$.

Therefore, the point of intersection of l and m is $\left(-2, \frac{1}{2}\right)$.

(3) Obtain the coordinates of the point of intersection of lines m and n.

[Sol] m: $y = \frac{1}{2}x + \frac{3}{2}$ …①, n: $x = -2$ …②

From ① and ②, $x = -2$ and $y = \frac{1}{2}$.

Therefore, the point of intersection of m and n is $\left(-2, \frac{1}{2}\right)$.

(4) Obtain the coordinates of the point of intersection of lines n and l.

[Sol] n: $x = -2$ …①, l: $y = -\frac{3}{2}x - \frac{5}{2}$ …②

From ① and ②, $x = -2$ and $y = \frac{1}{2}$.

Therefore, the point of intersection of n and l is $\left(-2, \frac{1}{2}\right)$.

Terminology: The point at which the lines perpendicular to and bisecting the three sides of a triangle intersect is called the **circumcenter** of the triangle.

M 164 a Equations of Straight Lines 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. Given $\triangle ABC$ from M163b, with vertices $A(0, 4)$, $B(-6, 0)$ and $C(2, 0)$, solve the following exercises.

(1) Find the coordinates of the circumcenter K, the orthocenter H and the center of gravity G.

[Sol] K: $\left(-2, \frac{1}{2}\right)$ from M163b

H: the equation of the perpendicular line from B to AC is:

$y = \frac{1}{2}(x+6)$ ... ①

the equation of the perpendicular line from A to BC is:

$x = 0$ ... ②

From ① and ②, $x = 0$, $y = 3$

Therefore, the coordinates of H are $(0, 3)$.

G: $x = \frac{0-6+2}{3} = -\frac{4}{3}$, $y = \frac{4+0+0}{3} = \frac{4}{3}$

Therefore, the coordinates of G are $\left(-\frac{4}{3}, \frac{4}{3}\right)$.

Ans. K: $\left(-2, \frac{1}{2}\right)$, H: $(0, 3)$, G: $\left(-\frac{4}{3}, \frac{4}{3}\right)$

(2) Obtain the slopes of the straight lines GK and GH.

[Sol] Using K: $\left(-2, \frac{1}{2}\right)$, H: $(0, 3)$ and G: $\left(-\frac{4}{3}, \frac{4}{3}\right)$,

The slope of GK is $\frac{5}{4}$, and the slope of GH is $\frac{5}{4}$.

Note: GK and GH have equal slopes. Therefore, K, H and G all lie on the same line.

2. Given $\triangle ABC$ with vertices $A(2, 6)$, $B(-4, 0)$ and $C(4, 0)$, solve the following exercises.

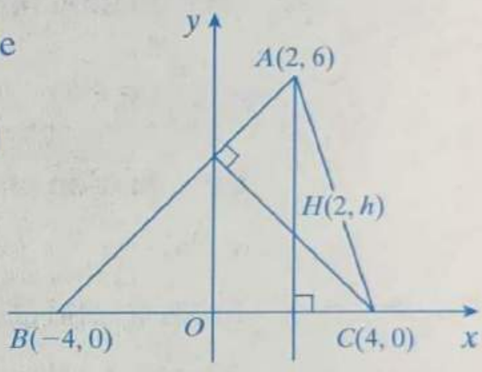
(1) Find the coordinates of the center of gravity G.

[Sol] $\dfrac{2-4+4}{3} = \dfrac{2}{3}$

$\dfrac{6+0+0}{3} = 2$

Ans. $G\left(\dfrac{2}{3}, 2\right)$

(2) Find the coordinates of the orthocenter H.



[Sol] Let $H(2, h)$ be the coordinates of the orthocenter.

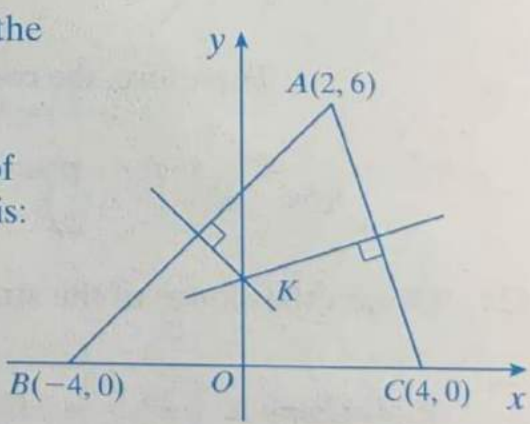
The product of the slope of AB and CH is -1.

$\dfrac{6-0}{2-(-4)} \cdot \dfrac{h-0}{2-4} = -1$

$\therefore h = 2$

Ans. $H(2, 2)$

(3) Find the coordinates of the circumcenter K.



[Sol] Let $K(0, k)$ be the coordinates of the circumcenter.

Since AB has slope 1, and its midpoint is $(-1, 3)$ the equation of the perpendicular bisector of AB is:

$$y - 3 = -(x + 1)$$
$$\therefore y = -x + 2$$

When $x = 0$, $y = 2$

Ans. $K(0, 2)$

(4) Find the equation of the line passing through G, H and K.

[Sol] Since the points are $G\left(\dfrac{2}{3}, 2\right)$, $H(2, 2)$ and $K(0, 2)$,

The equation of the line is $y = 2$.

M 165 a Equations of Straight Lines 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

1. Given point $P(1, 1)$ and line l: $3x + 4y = 12$, solve the following exercises.

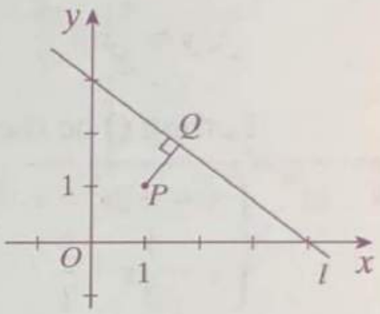
(1) Obtain the equation of a line m which passes through point P and is perpendicular to line l.

[Sol] Since line m is perpendicular to l, m has slope $\frac{4}{3}$.

Since m passes through $P(1, 1)$,

$$y-1=\frac{4}{3}(x-1)$$

$$\therefore\ y=\frac{4}{3}x-\frac{1}{3}\quad [4x-3y=1]$$



(2) Obtain the coordinates of the point of intersection Q of l and m.

[Sol]

$$\begin{cases} 3x+4y=12 & \cdots ① \\ y=\frac{4}{3}x-\frac{1}{3} & \cdots ② \end{cases}$$

From ① and ②, $x=\frac{8}{5}$, $y=\frac{9}{5}$

Therefore, the point of intersection of l and m is $Q\left(\frac{8}{5}, \frac{9}{5}\right)$.

(3) Obtain the length of PQ.

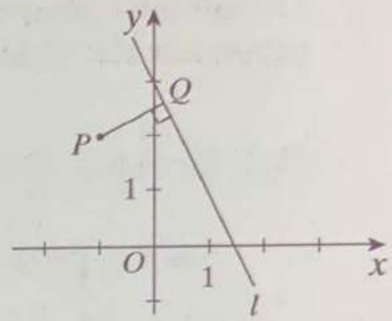
[Sol] Using $P(1, 1)$ and $Q\left(\frac{8}{5}, \frac{9}{5}\right)$,

$$PQ=\sqrt{\left(1-\frac{8}{5}\right)^2+\left(1-\frac{9}{5}\right)^2}=1$$

2. Obtain the length of the perpendicular dropped from point $P(-1, 2)$ to the line l: $y = -2x + 3$.

[Sol] Letting m be the perpendicular line dropped from point $P(-1, 2)$ to line l,

Since line m is perpendicular to l, m has slope $\frac{1}{2}$.



Since m passes through $P(-1, 2)$,

$$y - 2 = \frac{1}{2}(x + 1)$$

$$\therefore\ y = \frac{1}{2}x + \frac{5}{2} \quad [x - 2y = -5]$$

Letting Q be the point of intersection of l and m,

$$\begin{cases} y = -2x + 3 & \ldots ① \\ y = \frac{1}{2}x + \frac{5}{2} & \ldots ② \end{cases}$$

From ① and ②, $x = \frac{1}{5}$, $y = \frac{13}{5}$

Therefore, the point of intersection of l and m is $Q\left(\frac{1}{5}, \frac{13}{5}\right)$.

Using $P(-1, 2)$ and $Q\left(\frac{1}{5}, \frac{13}{5}\right)$,

$$PQ = \sqrt{\left(-1 - \frac{1}{5}\right)^2 + \left(2 - \frac{13}{5}\right)^2} = \frac{3\sqrt{5}}{5}$$

Therefore, the length of the perpendicular dropped from point P to line l is $\frac{3\sqrt{5}}{5}$.

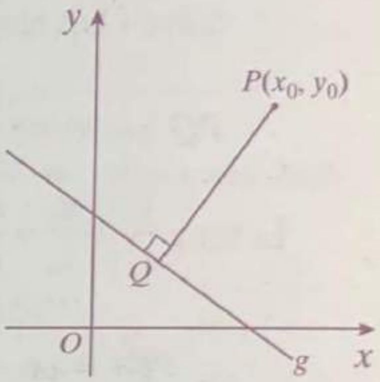
M 166 a Equations of Straight Lines 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1~

1. Show that the length of the perpendicular dropped from point $P(x_0, y_0)$ to line g: $y = mx + n \quad (m \neq 0)$ is $\dfrac{|mx_0 - y_0 + n|}{\sqrt{m^2+1}}$.

[Sol] g: $y = mx + n \quad (m \neq 0)$ $\quad \dots ①$



Let the perpendicular dropped from P touch g at Q.

The equation of line PQ is:

$$y - y_0 = \boxed{-\frac{1}{m}}(x - x_0) \quad \dots ②$$

From ① and ②, the coordinates of Q are:

$$x = \boxed{\frac{x_0 + my_0 - mn}{m^2+1}},\ y = \boxed{\frac{mx_0 + m^2y_0 + n}{m^2+1}} \quad \dots ③$$

Therefore, the length of PQ is:

$$PQ = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$$= \sqrt{\frac{m^2+1}{m^2}(x - x_0)^2} \quad \text{(from ②)}$$

$$= \sqrt{\frac{m^2+1}{m^2}\left(\frac{x_0 + my_0 - mn}{m^2+1} - x_0\right)^2} \quad \text{(from ③)}$$

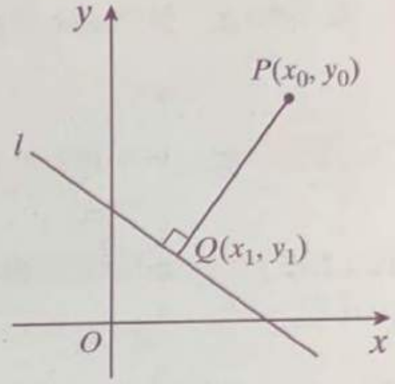
$$= \frac{|mx_0 - y_0 + n|}{\sqrt{m^2+1}}$$

2. Obtain the length of the perpendicular dropped from point $P(x_0, y_0)$ to the given line l using the method below.

$$l:\quad ax + by + c = 0 \qquad \ldots ①$$

[Sol] Letting the perpendicular dropped from P touch l at $Q(x_1, y_1)$,
Substituting this into ①,

$$ax_1 + by_1 + c = 0 \quad \ldots ①'$$



(a) When $ab \neq 0$,

Since l has slope $-\dfrac{a}{b}$,

PQ has slope $\dfrac{b}{a}$.

Letting $\dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{b}{a}$, $\dfrac{y_1 - y_0}{b} = \dfrac{x_1 - x_0}{a} = k$

$$y_1 - y_0 = bk, \quad x_1 - x_0 = ak \qquad \ldots ②$$

$$\therefore PQ^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2 = (a^2 + b^2)k^2 \qquad \ldots ③$$

From ②, $x_1 = \boxed{x_0 + ak}$ and $y_1 = \boxed{y_0 + bk}$

Substituting these values into ①',

$$a(x_0 + ak) + b(y_0 + bk) + c = 0$$

$$\therefore k = \boxed{-\frac{ax_0 + by_0 + c}{a^2 + b^2}} \qquad \ldots ④$$

Substituting ④ into ③,

$$PQ^2 = \boxed{\frac{(ax_0 + by_0 + c)^2}{a^2 + b^2}}$$

$$PQ = \boxed{\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}} \qquad \ldots ⑤$$

(b) When $ab = 0$,

If $a = 0$ and $b \neq 0$, then $x_0 = x_1$. $\therefore PQ = |y_1 - y_0| = \left|-\dfrac{c}{b} - y_0\right|$

If $b = 0$ and $a \neq 0$, then $y_0 = y_1$. $\therefore PQ = |x_1 - x_0| = \left|-\dfrac{c}{a} - x_0\right|$

(Note that these values are equal to those obtained when $a = 0$ and $b = 0$ are substituted into ⑤.)

From (a) and (b), $PQ = \boxed{\dfrac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}}$

M 167 a Equations of Straight Lines 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

Formula

> The length of the perpendicular dropped from point $P(x_0, y_0)$ to line l: $ax + by + c = 0$ is expressed as:
> $$\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

1. Using the above formula, find the distance between the origin and each of the following lines.

(1) $3x + 4y = 5$

[Sol] $\dfrac{|3\times 0 + 4\times 0 - 5|}{\sqrt{3^2+4^2}} = 1$

(2) $y = 2x - 3$

[Sol] $\dfrac{|2\times 0 - 0 - 3|}{\sqrt{2^2+(-1)^2}} = \dfrac{3\sqrt{5}}{5}$

(3) $3x + 4y - 6 = 0$

[Sol] $\dfrac{|3\times 0 + 4\times 0 - 6|}{\sqrt{3^2+4^2}} = \dfrac{6}{5}$

(4) $ax + by = c$

[Sol] $\dfrac{|a\times 0 + b\times 0 - c|}{\sqrt{a^2+b^2}} = \dfrac{|c|}{\sqrt{a^2+b^2}}$

2. Find the distance between point $(-5, 2)$ and each of the following lines.

(1) $3x + 8y = 2$

[Sol] $\dfrac{|3\times(-5) + 8\times 2 - 2|}{\sqrt{3^2+8^2}} = \dfrac{\sqrt{73}}{73}$

(2) $x + 2y + 1 = 0$

[Sol] $\dfrac{|-5 + 2\times 2 + 1|}{\sqrt{1^2+2^2}} = 0$

(3) $ax + by + c = 0$

[Sol] $\dfrac{|a\times(-5) + b\times 2 + c|}{\sqrt{a^2+b^2}} = \dfrac{|-5a + 2b + c|}{\sqrt{a^2+b^2}}$

3. Obtain the coordinates of point P that satisfy the following conditions.

(1) P is on the x-axis and at a distance of $\sqrt{5}$ units from line $y = 2x + 1$.

[Sol] Letting P be at $P(p, 0)$,

$$\sqrt{5}=\frac{|2p+1|}{\sqrt{2^2+1^2}}$$

$$\therefore |2p+1|=5$$

$$2p + 1 = \pm 5$$

$$p = 2, -3$$

$$\therefore P(2, 0), (-3, 0)$$

(2) P is on the y-axis and at a distance of 2 units from line $\frac{x}{2}+\frac{y}{3}=1$.

[Sol] Letting P be at $P(0, p)$,

$$2=\frac{\left|\frac{p}{3}-1\right|}{\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{3}\right)^2}}$$

$$\therefore \left|\frac{p}{3}-1\right|=\frac{\sqrt{13}}{3}$$

From $\frac{p}{3}-1=\pm\frac{\sqrt{13}}{3}$,

$$p=3\pm\sqrt{13}$$

$$\therefore P(0, 3\pm\sqrt{13})$$

Equations of Straight Lines 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

1. Given the straight lines l: $y=\frac{2}{3}x+\frac{1}{3}$ and m: $y=2x+1$, obtain the equation of the straight line that is symmetric to line l with respect to line m, by taking the following steps.

(1) Obtain the coordinates of point A which is the point of intersection of lines l and m.

[Sol] l: $y=\frac{2}{3}x+\frac{1}{3}$ …①
m: $y=2x+1$ …②
From ① and ②, $x=-\frac{1}{2}$ and $y=0$.
Therefore, the coordinates of point A are $\left(-\frac{1}{2}, 0\right)$.

(2) Obtain the coordinates of point B which is symmetric to point $C(1, 1)$ on line l with respect to line m.

[Sol] The equation of the line that is perpendicular to m: $y=2x+1$ …① and passes through point $C(1, 1)$ is:

$$y-1=-\frac{1}{2}(x-1)$$
$$y=-\frac{1}{2}x+\frac{3}{2} \quad \dots②$$

(**Note:** C can be any point on line l. In this case, we selected $C(1, 1)$. The same results would be obtained by selecting $\left(-1, -\frac{1}{3}\right)$, $\left(2, \frac{5}{3}\right)$, or any other point on line l.)

The coordinates of the point of intersection of ① and ② are $\left(\frac{1}{5}, \frac{7}{5}\right)$.
Letting the coordinates of point B be (X, Y),

$$\frac{X+1}{2}=\frac{1}{5}, \quad \frac{Y+1}{2}=\frac{7}{5}$$
$$\therefore X=-\frac{3}{5}, \quad Y=\frac{9}{5}$$

Ans. $B\left(-\frac{3}{5}, \frac{9}{5}\right)$

(3) Obtain the equation of a line that passes through points A and B.

[Sol] Using $A\left(-\frac{1}{2}, 0\right)$ and $B\left(-\frac{3}{5}, \frac{9}{5}\right)$,

The line that passes through point A and B has slope $=\dfrac{0-\frac{9}{5}}{-\frac{1}{2}+\frac{3}{5}}=-18$

Using point A and the slope, the equation of the line is:

$$y-0=-18\left(x+\frac{1}{2}\right)$$
$$\therefore y=-18\left(x+\frac{1}{2}\right) \quad [y=-18x-9]$$

2. Obtain the equation of the straight line that is symmetric to straight line l: $x - y - 2 = 0$ with respect to straight line m: $x + 2y + 1 = 0$.

[Sol] Letting A be the point of intersection of lines l and m,

Determining the coordinates of A,

l: $x - y - 2 = 0$ ...①

m: $x + 2y + 1 = 0$ ...②

From ① and ②, $x = 1$ and $y = -1$.

Therefore, the coordinates of point A are $(1, -1)$.

Since point $(2, 0)$ is on line l, let C be point $(2, 0)$

Letting B be the point symmetric to point $C(2, 0)$ on line l with respect to line m,

The equation of the line that is perpendicular to m: $x + 2y + 1 = 0$ ...②
and passes through point $C(2, 0)$ is $y - 0 = 2(x - 2)$

$$y = 2x - 4 \qquad \text{...③}$$

The coordinates of the point of intersection of ② and ③ are $\left(\frac{7}{5}, -\frac{6}{5}\right)$.

Letting the coordinates of point B be (X, Y),

$$\frac{X+2}{2} = \frac{7}{5}, \quad \frac{Y+0}{2} = -\frac{6}{5}$$

$$X = \frac{4}{5}, \qquad Y = -\frac{12}{5} \qquad \therefore B\left(\frac{4}{5}, -\frac{12}{5}\right)$$

Using $A(1, -1)$ and $B\left(\frac{4}{5}, -\frac{12}{5}\right)$,

The line that passes through point A and B has slope $= \dfrac{-\frac{12}{5}+1}{\frac{4}{5}-1} = 7$

Using point A and the slope, the equation of the line is:

$$y + 1 = 7(x - 1)$$

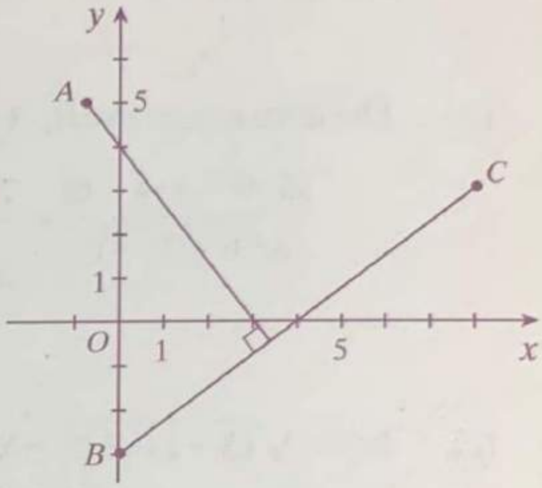
$$y = 7x - 8$$

M 169 a Equations of Straight Lines 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

1. Given the three points $A(-1, 5)$, $B(0, -3)$ and $C(8, 3)$, solve the following exercises.



(1) Obtain the equation of line BC.

[Sol] $y-(-3)=\dfrac{3-(-3)}{8-0}(x-0)$

$\therefore\ y=\dfrac{3}{4}x-3$

$[3x-4y-12=0]$

(2) Find the distance from A to line BC.

[Sol] The distance between point $A(-1, 5)$ and line $y=\dfrac{3}{4}x-3$ is:

$$\frac{\left|\dfrac{3}{4}\cdot(-1)-5-3\right|}{\sqrt{\left(\dfrac{3}{4}\right)^2+(-1)^2}}=7$$

(3) Using the result of exercise (2) above, find the area of $\triangle ABC$.

[Sol] The length of BC is:

$BC=\sqrt{(8-0)^2+(3+3)^2}=10$

Using the result of exercise (2),

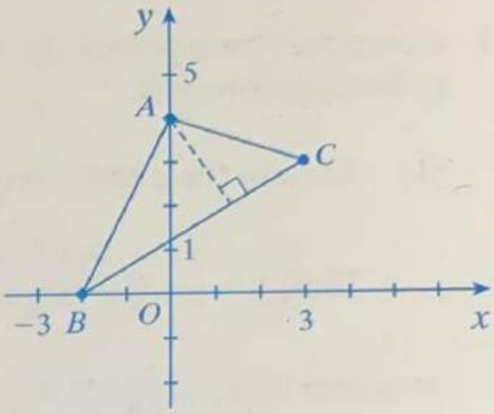
The area of $\triangle ABC$ is: $\dfrac{1}{2}\times10\times7=35$

M 169 b

2. Obtain the area of $\triangle ABC$ with vertices $A(0, 4)$, $B(-2, 0)$ and $C(3, 3)$, by applying the method from side a.

(a) The equation of BC is:

$3x - 5y + 6 = 0$



(b) The distance between A and BC is:

$$\frac{|3 \cdot 0 - 5 \cdot 4 + 6|}{\sqrt{3^2 + (-5)^2}} = \frac{7\sqrt{34}}{17}$$

(c) $BC = \sqrt{(3+2)^2 + 3^2} = \sqrt{34}$

Using the result of part (b), the area of $\triangle ABC$ is:

$$\frac{1}{2} \cdot \sqrt{34} \cdot \frac{7\sqrt{34}}{17} = 7$$

M 170 a Equations of Straight Lines 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1~

1. Obtain the coordinates of the center of gravity of the triangle formed by the three straight lines $x + 2y + 8 = 0$, $11x - y - 27 = 0$ and $7x - 9y + 33 = 0$.

[Sol] Determining the vertices,

$$\begin{cases} x + 2y + 8 = 0 & \dots① \\ 11x - y - 27 = 0 & \dots② \end{cases}$$

From ① and ②, one vertex is $(2, -5)$.

$$\begin{cases} 11x - y - 27 = 0 & \dots② \\ 7x - 9y + 33 = 0 & \dots③ \end{cases}$$

From ② and ③, one vertex is $(3, 6)$.

$$\begin{cases} x + 2y + 8 = 0 & \dots① \\ 7x - 9y + 33 = 0 & \dots③ \end{cases}$$

From ① and ③, one vertex is $(-6, -1)$.

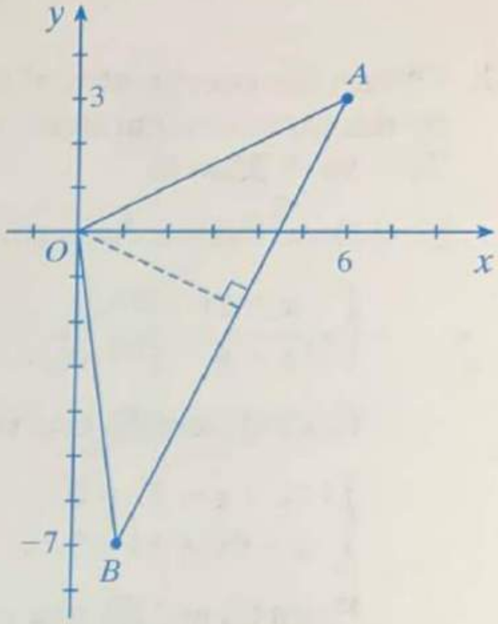
Letting G be the center of gravity, using the three vertices,

$$G = \left(\frac{2+3-6}{3}, \frac{-5+6-1}{3}\right)$$

$$\therefore G\left(-\frac{1}{3}, 0\right)$$

2. Obtain the area, S, of the triangle whose vertices are $O(0, 0)$, $A(6, 3)$ and $B(1, -7)$.

[Sol] Side AB has slope $\dfrac{-7-3}{1-6} = \dfrac{10}{5} = 2$



Using A and the slope, the equation of line AB is:

$$y - 3 = 2(x - 6)$$

$$\therefore y = 2x - 9 \quad [2x - y - 9 = 0]$$

Calculating the distance from $O(0, 0)$, to line AB: $2x - y - 9 = 0$

$$\frac{|2 \cdot 0 + (-1) \cdot 0 - 9|}{\sqrt{2^2 + (-1)^2}} = \frac{9}{\sqrt{5}} = \frac{9\sqrt{5}}{5}$$

Calculating the length of AB,

$$AB = \sqrt{(6-1)^2 + (3+7)^2} = 5\sqrt{5}$$

The area of $\triangle ABO$ is: $\dfrac{1}{2} \times 5\sqrt{5} \times \dfrac{9\sqrt{5}}{5} = \dfrac{45}{2}$

M 171 a Equations of Straight Lines 3

Time : to : Date Name

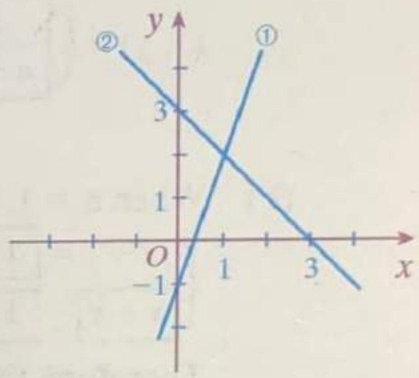
100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

1. In each of the following exercises, write the solution to the given equations. If there is no solution write "no solution". If there is an infinite number of solutions, write "indeterminable". Then, graph both lines and state the relationship between the lines as "intersect", "parallel" or "overlap".

(1) $\begin{cases} 3x - y - 1 = 0 & \ldots ① \\ x + y - 3 = 0 & \ldots ② \end{cases}$

Solution: (1, 2)

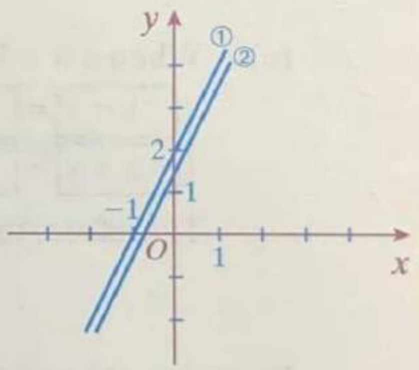
Relationship: (intersect)



(2) $\begin{cases} 2x - y + 2 = 0 & \ldots ① \\ 4x - 2y + 3 = 0 & \ldots ② \end{cases}$

Solution: (no solution)

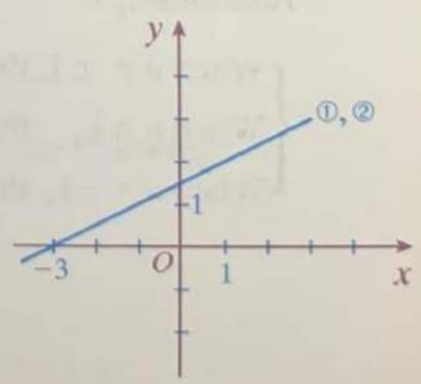
Relationship: (parallel)



(3) $\begin{cases} x - 2y + 3 = 0 & \ldots ① \\ 2x - 4y + 6 = 0 & \ldots ② \end{cases}$

Solution: (indeterminable)

Relationship: (overlap)



2. Solve the following equations, and state the relationship between the lines.

$$\begin{cases} x + ay = 1 & \cdots① \\ ax + y = 1 & \cdots② \end{cases}$$

[Sol] From ① × a − ②,

$$\boxed{(a^2-1)}\, y = \boxed{a-1}$$

(a) When $a \neq \pm 1$,

$$(x, y) = \left(\boxed{\frac{1}{a+1}}, \boxed{\frac{1}{a+1}}\right)$$

(b) When $a = 1$, equations ① and ② become:

$$\begin{cases} x + y = \boxed{1} \\ \boxed{x+y} = \boxed{1} \end{cases}$$

Therefore, the solution is indeterminable.

(c) When $a = -1$, equations ① and ② become:

$$\begin{cases} \boxed{x-y} = \boxed{1} \\ \boxed{-x+y} = \boxed{1} \end{cases}$$

Therefore, there is no solution.

From (a), (b) and (c), the two lines have the following relationships:

When $a \neq \pm 1$, the lines $\boxed{\text{intersect}}$.
When $a = 1$, the lines $\boxed{\text{overlap}}$.
When $a = -1$, the lines $\boxed{\text{are parallel}}$.

M 172 a Equations of Straight Lines 3

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1~

Solve the following equations, and state the relationship between the lines.

(1) $\begin{cases} ax + y = 1 & \dots ① \\ 4x + ay = 2 & \dots ② \end{cases}$

[Sol] From ① × a − ②,

$$(a^2 - 4)x = a - 2$$

(a) When $a \neq \pm 2$,

$$(x, y) = \left(\frac{1}{a+2}, \frac{2}{a+2}\right)$$

(b) When $a = 2$, equations ① and ② become:

$$\begin{cases} 2x + y = 1 \\ 4x + 2y = 2 \Rightarrow 2x + y = 1 \end{cases}$$

Therefore, the solution is indeterminable.

(c) When $a = -2$, equations ① and ② become:

$$\begin{cases} -2x + y = 1 \\ 4x - 2y = 2 \Rightarrow 2x - y = 1 \end{cases}$$

Therefore, there is no solution.

From (a), (b) and (c), the two lines have the following relationships:

$$\begin{cases} \text{When } a \neq \pm 2, & \text{the lines intersect.} \\ \text{When } a = 2, & \text{the lines overlap.} \\ \text{When } a = -2, & \text{the lines are parallel.} \end{cases}$$

(2) $\begin{cases} ax + y = a^2 & \ldots ① \\ x + ay = 1 & \ldots ② \end{cases}$

[Sol] From ① × a − ②,

$$(a^2 - 1)x = a^3 - 1$$

(a) When $a \neq \pm 1$,

$$(x, y) = \left(\frac{a^2 + a + 1}{a + 1}, -\frac{a}{a + 1}\right)$$

(b) When $a = 1$, equations ① and ② become:

$$\begin{cases} x + y = 1 \\ x + y = 1 \end{cases}$$

Therefore, the solution is indeterminable.

(c) When $a = -1$, equations ① and ② become:

$$\begin{cases} -x + y = 1 \\ x - y = 1 \end{cases}$$

Therefore, there is no solution.

From (a), (b) and (c), the two lines have the following relationships:

$$\begin{cases} \text{When } a \neq \pm 1, & \text{the lines intersect.} \\ \text{When } a = 1, & \text{the lines overlap.} \\ \text{When } a = -1, & \text{the lines are parallel.} \end{cases}$$

M 173 a Equations of Straight Lines 3

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. In each exercise below, find the coordinates of a fixed point through which the given line will always pass regardless of the value of k.

Ex. $kx - y + 8k + 3 = 0$

[Sol] $k(x + 8) - (y - 3) = 0$

$(-8, 3)$

Collect and group together the k terms and the "non-k" terms.

(1) $kx + y - 4k = 0$

[Sol] $k(x - 4) + y = 0$

$(4, 0)$

(2) $2x + ky + 3k - 4 = 0$

[Sol] $k(y + 3) + (2x - 4) = 0$

$(2, -3)$

(3) $y + 5k = k(7 - 3x) - 2$

[Sol] $k(2 - 3x) - (y + 2) = 0$

$\left(\frac{2}{3}, -2\right)$

(4) $k(5 + 2x) - 3k = y - 2k - 6$

[Sol] $k(2x + 4) - (y - 6) = 0$

$(-2, 6)$

2. In each exercise below, find the coordinates of a fixed point through which the given line will always pass regardless of the value of k.

Ex. $(k-1)y + x - k + 3 = 0$

[Sol] $(x - y + 3) + k(y - 1) = 0$

$\begin{cases} y-1=0 \\ x-y+3=0 \end{cases}$ ← The fixed point (x, y) must satisfy these two equations simultaneously.

Solving the simultaneous equations, $y = 1$ and $x = -2$.

Therefore, the coordinates of the fixed point are $(-2, 1)$.

(1) $kx + (k-2)y = 1$

[Sol] $kx + ky - 2y - 1 = 0$

$k(x + y) - (2y + 1) = 0$

$\begin{cases} x+y=0 \\ 2y+1=0 \end{cases}$

$x = \frac{1}{2}$ and $y = -\frac{1}{2}$

Therefore, the coordinates of the fixed point are $\left(\frac{1}{2}, -\frac{1}{2}\right)$.

(2) $(3k-2)x + (4-k)y - 1 + 2k = 0$

[Sol] $3kx - 2x + 4y - ky - 1 + 2k = 0$

$k(3x - y + 2) - (2x - 4y + 1) = 0$

$\begin{cases} 3x-y+2=0 \\ 2x-4y+1=0 \end{cases}$

$x = -\frac{7}{10}$ and $y = -\frac{1}{10}$

Therefore, the coordinates of the fixed point are $\left(-\frac{7}{10}, -\frac{1}{10}\right)$.

KUMON

M 174 a

Equations of Straight Lines 3

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

If two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ intersect, then $(ax + by + c) + k(a'x + b'y + c') = 0$ is a line passing through their point of intersection.

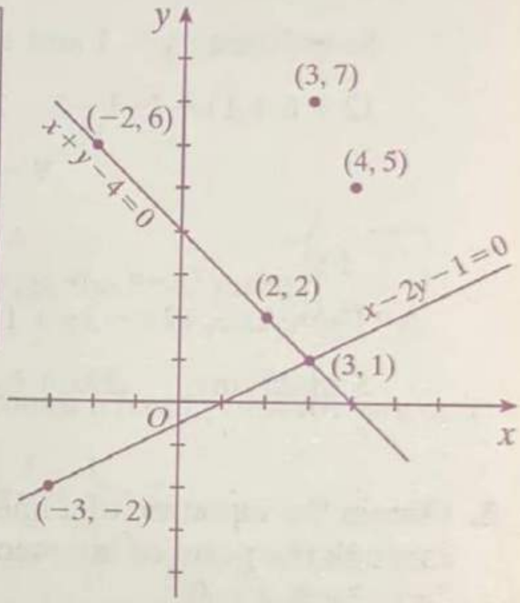
1. Given line $(x + y - 4) + k(x - 2y - 1) = 0$, in each exercise below, obtain the value of k at which the line will pass through the given point. If there is no such value, write "none".

Ex. (4, 5)

[Sol] $(4+5-4)+k(4-10-1)=0$

$5-7k=0$

$\therefore\ k=\frac{5}{7}$



(1) (3, 7)

[Sol] $(3+7-4)+k(3-14-1)=0$

$6-12k=0$

$\therefore\ k=\frac{1}{2}$

(2) (2, 2)

[Sol] $(2+2-4)+k(2-4-1)=0$

$-3k=0$

$\therefore\ k=0$

(3) (−2, 6)

[Sol] $(-2+6-4)+k(-2-12-1)=0$

$-15k=0$

$\therefore\ k=0$

(4) (−3, −2)

[Sol] $(-3-2-4)+k(-3+4-1)=0$

The LHS becomes $-9 + k\cdot 0$, not 0.

$\therefore$ none

From exercise 1. of side a, we can conclude that if two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ intersect, then the equation $(ax + by + c) + k(a'x + b'y + c') = 0$ represents all the lines passing through the point of intersection of the two lines, except line:

$$a'x + b'y + c' = 0$$

2. Obtain the equation of a line that passes through point $(1, -2)$ and through the point of intersection of the lines $2x - 3y + 1 = 0$ and $3x + y - 2 = 0$.

[Sol] $(2x - 3y + 1) + k(3x + y - 2) = 0$

Substituting $x = 1$ and $y = -2$,

$$(2 + 6 + 1) + k(3 - 2 - 2) = 0$$

$$9 - k = 0$$

$$\therefore k = 9$$

Therefore, $(2x - 3y + 1) + 9(3x + y - 2) = 0$

Simplifying, $29x + 6y - 17 = 0$

3. Obtain the equation of a line that passes through point $(-3, 4)$ and through the point of intersection of the lines $3x + y + 1 = 0$ and $2x + 3y + 2 = 0$.

[Sol] $(3x + y + 1) + k(2x + 3y + 2) = 0$

Substituting $x = -3$ and $y = 4$,

$$(-9 + 4 + 1) + k(-6 + 12 + 2) = 0$$

$$-4 + 8k = 0$$

$$\therefore k = \frac{1}{2}$$

Therefore, $(3x + y + 1) + \frac{1}{2}(2x + 3y + 2) = 0$

Simplifying, $8x + 5y + 4 = 0$

M 175 a Equations of Straight Lines 3

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

1. Show that line $ax+2by=1$ always passes through a fixed point for any pair of values of a and b satisfying the equation $a+b=3$.

[Sol] Since $a=3-b$, Substitute this value into the original equation of the line.

$\boxed{(3-b)}\ x+2by=1$

$\boxed{3x-bx+2by-1}=0$

$b(\boxed{2y-x})+(\boxed{3x-1})=0$

$\begin{cases}\boxed{2y-x=0}\\ \boxed{3x-1=0}\end{cases}$

$x=\boxed{\dfrac{1}{3}},\ y=\boxed{\dfrac{1}{6}}$

Therefore, the line always passes through the fixed point $\boxed{\left(\dfrac{1}{3},\dfrac{1}{6}\right)}$.

2. Show that line $\dfrac{x}{a}+\dfrac{y}{b}=1$ always passes through a fixed point for any pair of values of a and b satisfying the equation $\dfrac{1}{a}+\dfrac{1}{b}=2$.

[Sol] Since $\dfrac{1}{a}+\dfrac{1}{b}=2$, $\quad \dfrac{1}{b}=2-\dfrac{1}{a}$

$\dfrac{1}{a}x+\left(2-\dfrac{1}{a}\right)y=1$

$x+(2a-1)y-a=0$

$a(2y-1)+(x-y)=0$

$\begin{cases}2y-1=0\\ x-y=0\end{cases}$

$x=\dfrac{1}{2},\ y=\dfrac{1}{2}$

Therefore, the line always passes through the fixed point $\left(\dfrac{1}{2},\dfrac{1}{2}\right)$.

3. Determine the value(s) of m at which the following three lines do not form a triangle.

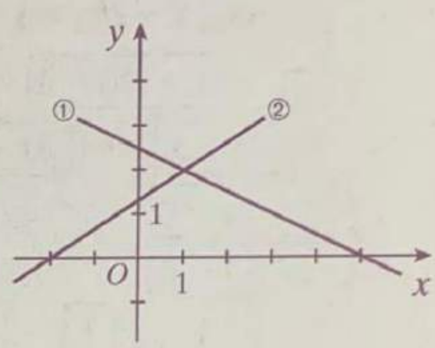
$$x + 2y = 5, \quad 2x - 3y = -4, \quad mx + y = 0$$

[Sol] $x + 2y = 5 \quad \ldots ①$

$2x - 3y = -4 \quad \ldots ②$

$mx + y = 0 \quad \ldots ③$

Since line ③ passes through the origin, lines ①, ② and ③ will not form a triangle in the following three cases:



(a) When ① and ③ are parallel.

$-\frac{1}{2} = -m, \ \therefore m = \frac{1}{2}$

(b) When ② and ③ are parallel.

$\frac{2}{3} = -m, \ \therefore m = -\frac{2}{3}$

(c) When ①, ② and ③ all intersect at one point.

Lines ① and ② intersect at point $(1, 2)$.
Line ③ passes through point $(1, 2)$ when $m = -2$.

Therefore, from (a), (b) and (c), $m = -2, -\frac{2}{3}, \frac{1}{2}$.

4. Determine the value(s) of a at which $(2 - a)x + 2y = 0$ and $2x - (1 + a)y = 0$ represent the same line.

[Sol] $\begin{cases} (2-a)x + 2y = 0 & \ldots ① \\ 2x - (1+a)y = 0 & \ldots ② \end{cases}$

Lines ① and ② pass through the origin.

(a) When $a \neq -1$,

① has slope $\frac{a-2}{2}$.

② has slope $\frac{2}{a+1}$.

Since lines ① and ② represent the same line,

$\frac{a-2}{2} = \frac{2}{a+1}$

$\therefore a = -2, 3$

(b) When $a = -1$,

Lines ① and ② become:

$\begin{cases} 3x + 2y = 0 & \ldots ①' \\ x = 0 & \ldots ②' \end{cases}$

Lines ①' and ②' do not represent the same line.

Therefore, from (a) and (b), $a = -2, 3$.

M 176 a Equations of Straight Lines 3

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1~

Ex. Obtain the equation of the line that passes through the point of intersection of lines $x+y=-1$ and $3x-2y=-1$, and is parallel to line $3x+2y=8$.

[Sol] Setting up the equation of the line we are looking for,

$$x+y+1+k(3x-2y+1)=0 \qquad \ldots ①$$

Rearranging ①,

$$\boxed{(1+3k)}\,x+\boxed{(1-2k)}\,y+\boxed{(1+k)}=0 \qquad \ldots ②$$

Since line ② and $3x+2y=8$ are parallel,

$$\boxed{\frac{3k+1}{2k-1}}=\boxed{-\frac{3}{2}}$$

$$\therefore k=\boxed{\frac{1}{12}} \qquad \ldots ③$$

Substituting ③ into ① and simplifying,

$$\boxed{15x+10y+13}=0$$

Answers: $(1+3k)$, $(1-2k)$, $(1+k)$, $\frac{3k+1}{2k-1}$, $-\frac{3}{2}$, $\frac{1}{12}$, $15x+10y+13$

1. Obtain the equation of the line that passes through the point of intersection of lines $2x+y=-3$ and $x-2y=1$, and is parallel to line $5x+2y=5$.

[Sol] Setting up the equation of the line we are looking for,

$$2x+y+3+k(x-2y-1)=0 \qquad \ldots ①$$

Rearranging ①,

$$(2+k)x+(1-2k)y+(3-k)=0 \qquad \ldots ②$$

Since line ② and $5x+2y=5$ are parallel,

$$\frac{k+2}{2k-1}=-\frac{5}{2}$$

$$\therefore k=\frac{1}{12} \qquad \ldots ③$$

Substituting ③ into ① and simplifying,

$$5x+2y+7=0$$

2. Obtain the equation of the line that passes through the point of intersection of lines $x-2y=3$ and $2x+3y=5$, and is perpendicular to line $x+y=10$.

[Sol] Setting up the equation of the line we are looking for,

$$x-2y-3+k(2x+3y-5)=0 \qquad \cdots ①$$

Rearranging ①,

$$(1+2k)x-(2-3k)y-(3+5k)=0 \qquad \cdots ②$$

Since line ② and $x+y=10$ are perpendicular,

$$\frac{2k+1}{2-3k}\cdot(-1)=-1$$

$$\therefore k=\frac{1}{5} \qquad \cdots ③$$

Substituting ③ into ① and simplifying,

$$7x-7y-20=0$$

3. Obtain the equation of the line that passes through the point of intersection of lines $x+3y=2$ and $4x+y=3$, and is perpendicular to line $2x-y=6$.

[Sol] Setting up the equation of the line we are looking for,

$$x+3y-2+k(4x+y-3)=0 \qquad \cdots ①$$

Rearranging ①,

$$(1+4k)x+(3+k)y-(2+3k)=0 \qquad \cdots ②$$

Since line ② and $2x-y=6$ are perpendicular,

$$-\frac{4k+1}{k+3}\cdot(-1)=\frac{1}{2}$$

$$\therefore k=\frac{1}{7} \qquad \cdots ③$$

Substituting ③ into ① and simplifying,

$$11x+22y-17=0$$

Equations of Straight Lines 3

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

1. Obtain the equation of the following straight lines.

(1) A line passing through point $(-4, 3)$ with slope 2.

[Sol] $y-3=2(x+4)$

$\therefore y=2x+11 \qquad [2x-y+11=0]$

(2) A line passing through points $(-1, 3)$ and $(5, -6)$.

[Sol] $m=\dfrac{-6-3}{5+1}=-\dfrac{9}{6}=-\dfrac{3}{2}$

$y-3=-\dfrac{3}{2}(x+1)$ ⟵ Substituted point $(-1, 3)$

$\therefore y=-\dfrac{3}{2}x+\dfrac{3}{2} \qquad [3x+2y-3=0]$

(3) A line passing through points $(2, 0)$ and $(0, -4)$.

[Sol] $\dfrac{x}{2}+\dfrac{y}{(-4)}=1$

$\therefore \dfrac{x}{2}-\dfrac{y}{4}=1 \qquad [y=2x-4 \text{ or } 2x-y-4=0]$

(4) A line passing through point $(1, -5)$ and parallel to line $4x+y-1=0$.

[Sol] The slope of $4x+y-1=0$ is -4.

$y+5=-4(x-1)$

$\therefore y=-4x-1 \qquad [4x+y+1=0]$

(5) A line passing through point $(-3, -2)$ and perpendicular to the line passing through points $(-1, 4)$ and $(5, -5)$.

[Sol] The slope of the line passing through $(-1, 4)$ and $(5, -5)$ is

$m_1=\dfrac{-5-4}{5+1}=-\dfrac{9}{6}=-\dfrac{3}{2}.$

The slope of the line perpendicular to the line passing through $(-1, 4)$ and $(5, -5)$ is $m_2=\dfrac{2}{3}$.

Using point $(-3,-2)$ and slope m_2, $\quad y+2=\dfrac{2}{3}(x+3)$

$y=\dfrac{2}{3}x$

M 177 b

2. Find the value(s) of k at which the given two lines are parallel.

(1) $2y-5kx+6=0,\quad 3y+x-1=0$

[Sol] $2y-5kx+6=0$ has slope $m_1=\frac{5}{2}k$. $3y+x-1=0$ has slope $m_2=-\frac{1}{3}$.

The two lines would be parallel if and only if, $m_1=m_2$.

$$\frac{5}{2}k=-\frac{1}{3}$$

$$k=-\frac{2}{15}$$

(2) $4y-4k^2x-5=0,\quad y-(k+6)x+3=0$

[Sol] $4y-4k^2x-5=0$ has slope $m_1=k^2$.

$y-(k+6)x+3=0$ has slope $m_2=k+6$.

The two lines would be parallel if and only if, $m_1=m_2$.

$$k^2=k+6$$

$$k^2-k-6=0$$

$$(k+2)(k-3)=0$$

$$k=-2,3$$

3. Find the value of k at which the given two lines are perpendicular.

(1) $3y-7kx-4=0,\quad 7y+x+2=0$

[Sol] $3y-7kx-4=0$ has slope $m_1=\frac{7}{3}k$. $7y+x+2=0$ has slope $m_2=-\frac{1}{7}$.

The two lines would be perpendicular if and only if, $m_1m_2=-1$.

$$\frac{7}{3}k\cdot\left(-\frac{1}{7}\right)=-1$$

$$k=3$$

(2) $4y-\frac{k}{2}x+1=0,\quad 2y+4x-3=0$

[Sol] $4y-\frac{k}{2}x+1=0$ has slope $m_1=\frac{k}{8}$. $2y+4x-3=0$ has slope $m_2=-2$.

The two lines would be perpendicular if and only if, $m_1m_2=-1$.

$$\frac{k}{8}\cdot(-2)=-1$$

$$k=4$$

M 178 a Equations of Straight Lines 3

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

1. Given points $A(-1, -4)$ and $B(3, 5)$, obtain the equation of the line that is perpendicular to and bisecting line segment AB.

[Sol] The midpoint of AB is $M\left(1, \frac{1}{2}\right)$.

The slope of AB is $\frac{9}{4}$.

Therefore, the equation of the line is $y - \frac{1}{2} = -\frac{4}{9}(x-1)$.

$\therefore\ y = -\frac{4}{9}x + \frac{17}{18}$ $[8x + 18y - 17 = 0]$

2. Obtain the value of a, at which the three given points all lie on the same line.

$$A(-3, -2),\quad B(1, 6),\quad C(a, 4)$$

[Sol] In order for the three points to lie on the same line, the slope of AB must equal the slope of BC.

The slope of AB is 2 ...① The slope of BC is $-\frac{2}{a-1}$ ...②

Since ① and ② must be equal, $2 = -\frac{2}{a-1}$

$2a - 2 = -2$

$a = 0$

3. Obtain the coordinates of point B which is symmetric to point $A(5, 4)$ with respect to line $3x - 2y + 6 = 0$.

[Sol] The equation of the line perpendicular to $3x - 2y + 6 = 0$ ...①

which passes through point $A(5, 4)$ is $y - 4 = -\frac{2}{3}(x-5)$ ...②

The coordinates of the point of intersection of ① and ② are $(2, 6)$.

Letting the coordinates of point B be (X, Y),

$\frac{X+5}{2} = 2,\quad \frac{Y+4}{2} = 6$

$\therefore\ X = -1,\quad Y = 8$ Ans. $B(-1, 8)$

4. Obtain the equation of a straight line that is symmetric to line l : $y = x - 3$ with respect to line m : $y = -\frac{1}{3}x - \frac{1}{3}$.

[Sol] Letting A be the point of intersection of lines l and m,
Determining the coordinates of A,

l: $y = x - 3$ ...①

m: $y = -\frac{1}{3}x - \frac{1}{3}$ ...②

From ① and ②, $x = 2$ and $y = -1$.
Therefore, the coordinates of point A are $(2, -1)$.
Since point $(0, -3)$ is on line l, let C be point $(0, -3)$.
Letting B be the point symmetric to point $C(0, -3)$ on line l with respect to line m,

The equation of the line that is perpendicular to m : $y = -\frac{1}{3}x - \frac{1}{3}$ ...②
and passes through point $C(0, -3)$ is $y + 3 = 3(x - 0)$

$$y = 3x - 3 \quad \text{...③}$$

The coordinates of the point of intersection of ② and ③ are $\left(\frac{4}{5}, -\frac{3}{5}\right)$.

Letting the coordinates of point B be (X, Y),

$$\frac{X+0}{2} = \frac{4}{5}, \quad \frac{Y-3}{2} = -\frac{3}{5}$$

$$X = \frac{8}{5}, \qquad Y = \frac{9}{5} \qquad \therefore B\left(\frac{8}{5}, \frac{9}{5}\right)$$

Using $A(2, -1)$ and $B\left(\frac{8}{5}, \frac{9}{5}\right)$,

The line that passes through points A and B has slope $= -7$
Using point A and the slope, the equation of the line is:

$$y + 1 = -7(x - 2)$$
$$y = -7x + 13$$

Equations of Straight Lines 3

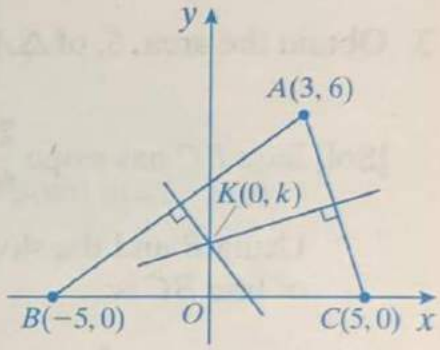
Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1~

1. Given $\triangle ABC$ with vertices $A(3, 6)$, $B(-5, 0)$ and $C(5, 0)$, find the coordinates of the circumcenter, K, the orthocenter, H, and the center of gravity, G, of the triangle.

[Sol] K: Let $K(0, k)$ be the coordinates of the circumcenter.

Since AB has slope $\frac{3}{4}$, and its midpoint is $(-1, 3)$, the equation of the perpendicular bisector of AB is:



$$y - 3 = -\frac{4}{3}(x + 1)$$

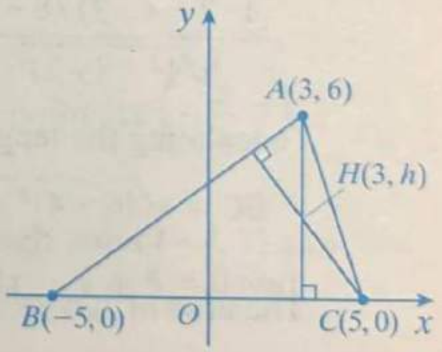
$$\therefore\ y = -\frac{4}{3}x + \frac{5}{3}$$

When $x = 0$, $y = \frac{5}{3}$

Therefore, $K\left(0, \frac{5}{3}\right)$

H: Let $H(3, h)$ be the coordinates of the orthocenter.

The product of the slopes of AB and CH is -1.



$$\frac{3}{4}\cdot\frac{h-0}{3-5} = -1$$

$$\therefore\ h = \frac{8}{3}$$

Therefore, $H\left(3, \frac{8}{3}\right)$

G: $\frac{3-5+5}{3} = 1$, $\quad \frac{6+0+0}{3} = 2$

Therefore, $G(1, 2)$

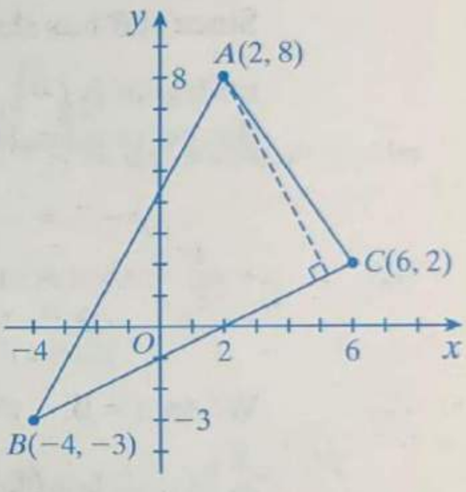
Answer: $K\left(0, \frac{5}{3}\right)$, $H\left(3, \frac{8}{3}\right)$, $G(1, 2)$

2. Find the distance between point $(-3, 2)$ and the line $6x + 5y = -1$.

[Sol] $\dfrac{|6\times(-3)+5\times2+1|}{\sqrt{6^2+5^2}}=\dfrac{7\sqrt{61}}{61}$

3. Obtain the area, S, of $\triangle ABC$ with vertices $A(2, 8)$, $B(-4, -3)$ and $C(6, 2)$.

[Sol] Side BC has slope $\dfrac{2+3}{6+4}=\dfrac{5}{10}=\dfrac{1}{2}$



Using B and the slope, the equation of line BC is:

$$y+3=\frac{1}{2}(x+4)$$

$$\therefore\ y=\frac{1}{2}x-1\quad [x-2y-2=0]$$

Calculating the distance from $A(2, 8)$ to line BC: $x-2y-2=0$,

$$\frac{|1\cdot2+(-2)\cdot8-2|}{\sqrt{1^2+(-2)^2}}=\frac{16}{\sqrt{5}}=\frac{16\sqrt{5}}{5}$$

Calculating the length of BC,

$$BC=\sqrt{(6+4)^2+(2+3)^2}=5\sqrt{5}$$

The area of $\triangle ABC$ is $\dfrac{1}{2}\times5\sqrt{5}\times\dfrac{16\sqrt{5}}{5}=40$

Equations of Straight Lines 3

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. In each exercise below, find the coordinates of a fixed point through which the given line always passes regardless of the value of k.

(1) $kx - 2y + 3k = 0$

[Sol] $k(x + 3) - 2y = 0$

$x = -3$ and $y = 0$

Therefore, the coordinates of the fixed point are $(-3, 0)$.

(2) $4kx + (k - 1)y = -3$

[Sol] $4kx + ky - y + 3 = 0$

$k(4x + y) - (y - 3) = 0$

$$\begin{cases} 4x + y = 0 \\ y - 3 = 0 \end{cases}$$

$x = -\dfrac{3}{4}$ and $y = 3$

Therefore, the coordinates of the fixed point are $\left(-\dfrac{3}{4}, 3\right)$.

2. Obtain the equation of a line that passes through point $(-4, 7)$ and through the point of intersection of the lines $2x - y + 5 = 0$ and $3x + 4y - 6 = 0$.

[Sol] $(2x - y + 5) + k(3x + 4y - 6) = 0$

Substituting $x = -4$ and $y = 7$,

$(-8 - 7 + 5) + k(-12 + 28 - 6) = 0$

$-10 + 10k = 0$

$k = 1$

Therefore, $(2x - y + 5) + (3x + 4y - 6) = 0$

Simplifying, $5x + 3y - 1 = 0$

3. Obtain the equation of the line that passes through the point of intersection of lines $4x - y = -1$ and $x + 6y = -3$, and is parallel to line $6x - 3y = 2$.

[Sol] Setting up the equation of the line we are looking for,

$$4x - y + 1 + k(x + 6y + 3) = 0 \qquad \cdots ①$$

Rearranging ①,

$$(4 + k)x + (6k - 1)y + (1 + 3k) = 0 \qquad \cdots ②$$

Since line ② and $6x - 3y = 2$ are parallel,

$$\frac{4+k}{1-6k} = 2$$

$$\therefore k = -\frac{2}{13} \qquad \cdots ③$$

Substituting ③ into ① and simplifying,

$$50x - 25y + 7 = 0$$

4. Obtain the equation of the line that passes through the point of intersection of lines $x - 4y = 3$ and $3x - 2y = 5$, and is perpendicular to line $5x + y = 6$.

[Sol] Setting up the equation of the line we are looking for,

$$x - 4y - 3 + k(3x - 2y - 5) = 0 \qquad \cdots ①$$

Rearranging ①,

$$(1 + 3k)x - 2(2 + k)\,y - (3 + 5k) = 0 \qquad \cdots ②$$

Since line ② and $5x + y = 6$ are perpendicular,

$$\frac{1+3k}{2(2+k)} = \frac{1}{5}$$

$$\therefore k = -\frac{1}{13} \qquad \cdots ③$$

Substituting ③ into ① and simplifying,

$$5x - 25y - 17 = 0$$

KUMON

M 181 a

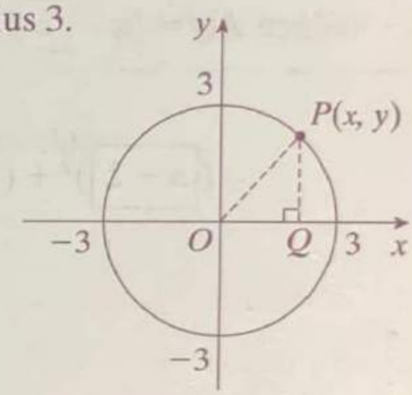
Equations of Circles

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

In each of the following exercises, obtain the equation of the circle described.

(1) A circle with center at the origin O and radius 3.



[Sol] Let $P(x, y)$ be any random point on the circumference of the circle where OP is the radius of the circle, and let Q be a point on the x-axis where PQ is perpendicular to the x-axis.

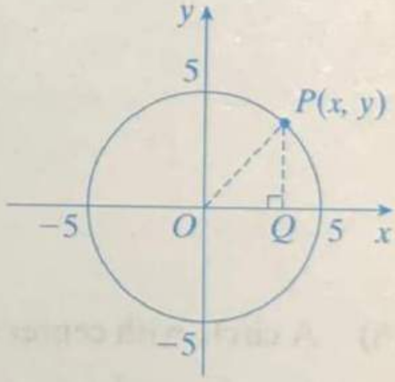
$\therefore OQ^2 + QP^2 = OP^2$

Substituting,

$\therefore x^2 + \boxed{y}^2 = \boxed{9}$

(2) A circle with center at the origin O and radius 5.

$x^2 + y^2 = 25$

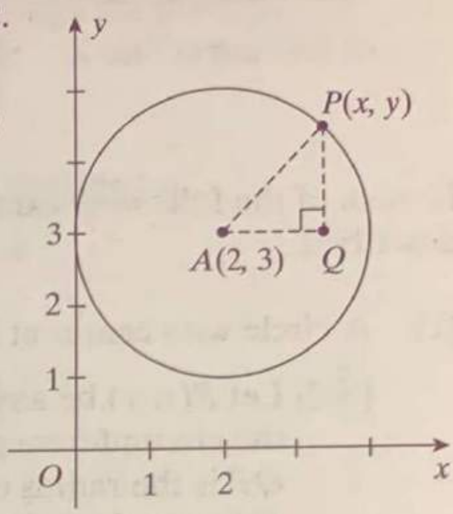


(3) A circle with center at the origin O and radius r.

$x^2 + y^2 = r^2$

(4) A circle with center at $A(2, 3)$ and radius 2.

[Sol] Let $P(x, y)$ be any random point on the circumference of the circle and Q be a point in the circle that creates a right triangle.



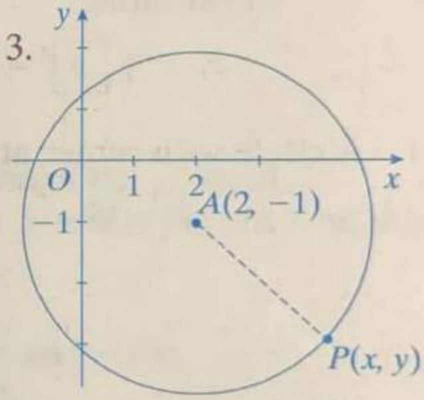
Thus, $AQ^2 + QP^2 = AP^2$

Since $AQ = (x - \boxed{2})$, $PQ = (y - 3)$,

$$\therefore (\boxed{x-2})^2 + (y-3)^2 = \boxed{4}$$

(5) A circle with center at $A(2, -1)$ and radius 3.

$(x-2)^2 + (y+1)^2 = 9$



(6) A circle with center at $A(a, b)$ and radius r.

$(x-a)^2 + (y-b)^2 = r^2$

KUMON

M 182 a Equations of Circles

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

Formulas

The equation of a circle with its:
- center at (0, 0) and radius r is: $x^2 + y^2 = r^2$
- center at (a, b) and radius r is: $(x - a)^2 + (y - b)^2 = r^2 \ (r > 0)$

1. Obtain the equation of each of the following circles.

(1) A circle with center at (0, 0) and radius 4.

$x^2 + y^2 = 16$

(2) A circle with center at (2, 1) and radius 2.

$(x-2)^2 + (y-1)^2 = 4$

(3) A circle with center at $(3, -4)$ and radius 5.

$(x-3)^2 + (y+4)^2 = 25$

(4) A circle with center at (0, 5) and radius $\sqrt{3}$.

$x^2 + (y-5)^2 = 3$

(5) A circle with center at $(-1, 3)$ and radius $2\sqrt{2}$.

$(x+1)^2 + (y-3)^2 = 8$

2. Obtain the equation and draw each of the following circles.

(1) A circle with center at $(3, 4)$ and passing through point $(1, 2)$.

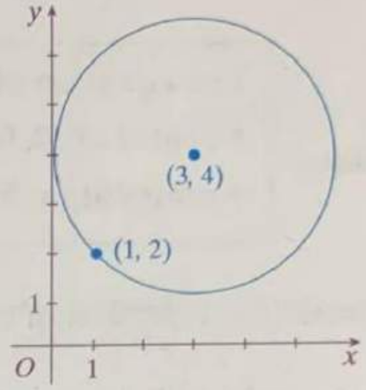
[Sol] Because the circle passes through $(1, 2)$, that point lies on the circle's circumference.

Using the distance formula to calculate the length of the radius,

$$\therefore r = \sqrt{(3-1)^2 + (4-2)^2} = 2\sqrt{2}$$

$\therefore$ The equation is:

$$(x-3)^2 + (y-4)^2 = 8$$



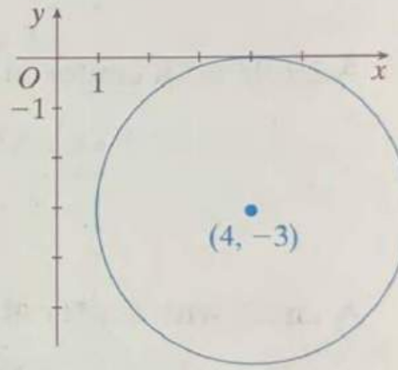
(2) A circle with center at $(4, -3)$ and touching the x-axis at only 1 point.

[Sol] Since the circle touches the x-axis at only one point, the point is directly above the center. Thus, it touches $(4, 0)$.

$\therefore r = 3$

$\therefore$ The equation is:

$$(x-4)^2 + (y+3)^2 = 9$$



(3) A circle with $(1, 1)$ and $(7, 8)$ at the endpoints of its diameter. (A diameter is a line segment that passes through the center of a circle and whose endpoints lie on the circle.)

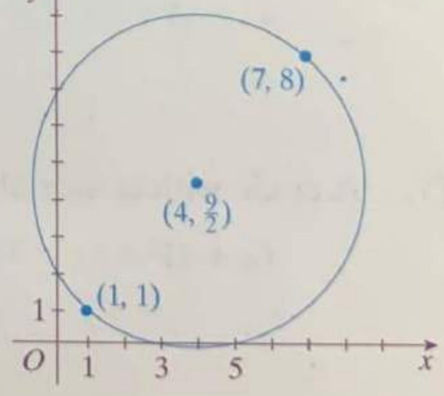
[Sol] Using the midpoint formula [M143b], we know that the center is at:

$$\left(\frac{1+7}{2}, \frac{1+8}{2}\right) = \left(4, \frac{9}{2}\right)$$

$$r = \sqrt{(4-1)^2 + \left(\frac{9}{2} - 1\right)^2} = \frac{\sqrt{85}}{2}$$

$\therefore$ The equation is:

$$(x-4)^2 + \left(y - \frac{9}{2}\right)^2 = \frac{85}{4}$$



M 183 a

Equations of Circles

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. Rewrite each of the following equations in the form $(x-a)^2+(y-b)^2=r^2$, and state the center and radius of the circle.

Ex. $x^2+y^2-2x-4y-20=0$

[Sol] Rearranging by completing the square,

$(x-1)^2+(y-2)^2=25$

$\therefore$ center: $(1, 2)$, radius: 5

(1) $x^2+y^2+6x-2y-6=0$

[Sol] Rearranging by completing the square,

$(x+3)^2+(y-1)^2=16$

$\therefore$ center: $(-3, 1)$, radius: 4

(2) $x^2+6x+y^2-4y=0$

[Sol] Rearranging by completing the square,

$(x+3)^2+(y-2)^2=13$

$\therefore$ center: $(-3, 2)$, radius : $\sqrt{13}$

(3) $3x^2+3y^2+10y=0$

[Sol] Rearranging by completing the square,

$$x^2+\left(y+\frac{5}{3}\right)^2=\frac{25}{9}$$

$\therefore$ center: $\left(0, -\frac{5}{3}\right)$, radius: $\frac{5}{3}$

M 183 b

2. For each of the following equations of circles, state the center and radius.

(1) $3x^2 + 3y^2 + 2x + y = 1$

[Sol] $x^2 + y^2 + \frac{2}{3}x + \frac{1}{3}y = \frac{1}{3}$

$$\left(x + \frac{1}{3}\right)^2 + \left(y + \frac{1}{6}\right)^2 = \frac{17}{36}$$

$\therefore$ center: $\left(-\frac{1}{3}, -\frac{1}{6}\right)$, radius: $\frac{\sqrt{17}}{6}$

(2) $x^2 + y^2 + ax + by + c = 0$, where $c < 0$

[Sol] $x^2 + ax + y^2 + by + c = 0$

$$\left(x^2 + ax + \frac{a^2}{4}\right) - \frac{a^2}{4} + \left(y^2 + by + \frac{b^2}{4}\right) - \frac{b^2}{4} + c = 0$$

$$\left(x + \frac{a}{2}\right)^2 + \left(y + \frac{b}{2}\right)^2 = \frac{a^2}{4} + \frac{b^2}{4} - c$$

$\therefore$ center: $\left(-\frac{a}{2}, -\frac{b}{2}\right)$, radius: $\sqrt{\frac{a^2}{4} + \frac{b^2}{4} - c}$

3. Depending on the values of a, b and c, the equation $x^2 + y^2 + 2ax + 2by + c = 0$ does not always represent a circle. Find the values of a, b and c which do not represent a circle.

(Hint) Recall the conditions of a circle.

[Sol] Remember that for a circle with the equation

$(x - a)^2 + (y - b)^2 = r^2, \quad r > 0. \qquad \therefore r^2 \boxed{>} 0.$

$x^2 + y^2 + 2ax + 2by + c = 0$

$(x^2 + 2ax + a^2) - a^2 + (y^2 + 2by + b^2) - b^2 + c = 0$ Rearranging

$(x + a)^2 + (y + b)^2 = a^2 + b^2 - c$

$\therefore x^2 + y^2 + 2ax + 2by + c = 0$ does not represent a circle when $\boxed{a^2 + b^2 - c \leq 0}$.

Equations of Circles

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

In each of the following exercises, draw a rough sketch, and obtain the equation of the given figure(s).

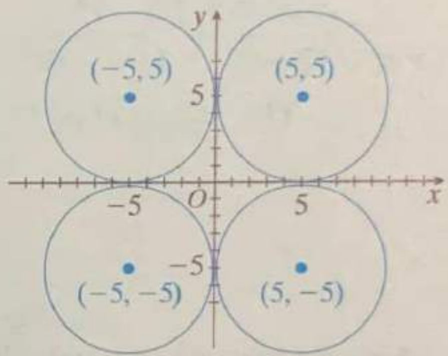
(1) The circles touching both coordinate axes with radius 5. (There are four.)

[Sol] Since the circles touch both the x-axis and y-axis, their centers must be: $(\pm 5, \mp 5)$, $(\pm 5, \pm 5)$

$\therefore$ The equations are:

$(x \pm 5)^2 + (y \pm 5)^2 = 25$

$(x \pm 5)^2 + (y \mp 5)^2 = 25$



(2) A circle with $(-2, 5)$ and $(6, -1)$ at the endpoints of a diameter.

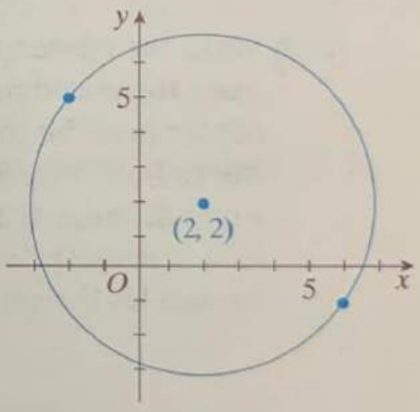
[Sol] Since the points create a diameter, the center lies in the middle. Using the midpoint formula,

$$\text{center} = \left(\frac{-2+6}{2}, \frac{5-1}{2}\right) = (2, 2)$$

$r = \sqrt{(2+2)^2 + (2-5)^2} = 5$

$\therefore$ The equation is:

$(x-2)^2 + (y-2)^2 = 25$



(3) A circle passing through $A(2, 3)$ and $B(-2, 1)$ with its center on the x-axis.

[Sol] Let the circle's center be $C = (a, 0)$.

$\therefore r = AC = BC$

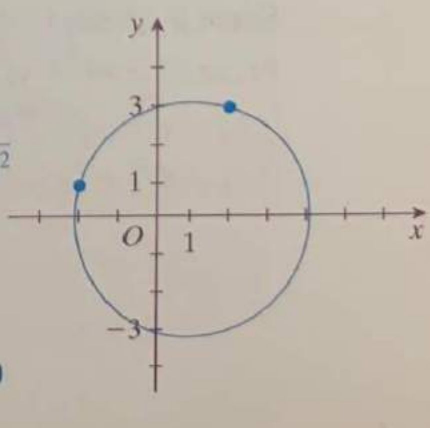
$= \sqrt{(a-2)^2 + 3^2} = \sqrt{(a+2)^2 + 1^2}$

$\therefore a = 1$

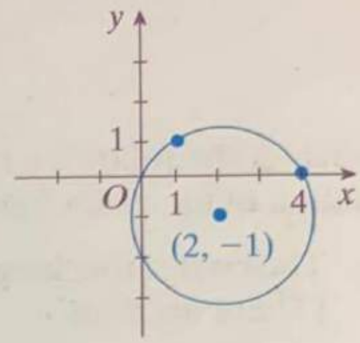
Thus, $C = (1, 0)$

$\therefore r = \sqrt{(2-1)^2 + 3^2} = \sqrt{10}$

$\therefore$ The equation is: $(x-1)^2 + y^2 = 10$



(4) A circle passing through the three points $(0, 0)$, $(4, 0)$ and $(1, 1)$.



[Sol] Let (a, b) be the center. Therefore,

$$\begin{cases} \sqrt{a^2+b^2} = \sqrt{(a-4)^2+b^2} \\ \sqrt{a^2+b^2} = \sqrt{(a-1)^2+(b-1)^2} \end{cases}$$

Solving these two equations results in:

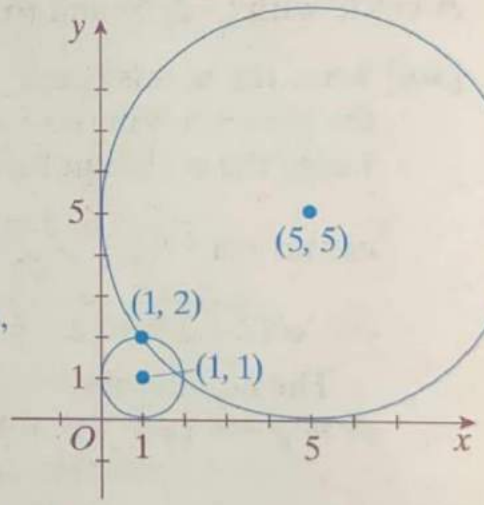
$a = 2$ and $b = -1$

$\therefore$ center: $(2, -1)$

$r = \sqrt{(2-0)^2+(-1-0)^2} = \sqrt{5}$

$\therefore$ The equation is: $(x-2)^2+(y+1)^2 = 5$

(5) A circle passing through $(1, 2)$ touching both axes. (There are two.)



[Sol] Since the circle touches both axes, the coordinates of the center must be the length of the radius, because it is the same distance from the x-axis as it is from the y-axis. Therefore, we can let the center be (r, r).

$\therefore$ The equation is: $(x-r)^2+(y-r)^2 = r^2$

Since it passes through $(1, 2)$,

From $(1-r)^2+(2-r)^2 = r^2$, $r = 1, 5$

$$\begin{cases} (x-1)^2+(y-1)^2 = 1 \\ (x-5)^2+(y-5)^2 = 25 \end{cases}$$

Equations of Circles

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

1. Obtain the equation of a circle passing through the three points $(1, -1)$, $(5, 1)$ and $(-2, 0)$.

[Sol] $x^2 + y^2 + ax + by + c = 0$

Substituting the three points,

substituting $(1, -1)$, $a - b + c = -2$

substituting $(5, 1)$, $\boxed{5a + b + c = -26}$

substituting $(-2, 0)$, $\boxed{-2a + c = -4}$

Solving for a, b and c, we can derive the equation.

$\therefore a = -2, b = -8, c = -8$

$\therefore$ The equation is: $x^2 + y^2 - 2x - 8y - 8 = 0$ [or $(x - 1)^2 + (y - 4)^2 = 25$]

2. Obtain the equation of a circle passing through the three points $(1, 1)$, $(3, 0)$ and $(1, -3)$.

[Sol] $x^2 + y^2 + ax + by + c = 0$

Substituting the three points,

substituting $(1, 1)$, $a + b + c = -2$

substituting $(3, 0)$, $3a + c = -9$

substituting $(1, -3)$, $a - 3b + c = -10$

Solving for a, b, and c, we can derive the equation.

$\therefore a = -\frac{5}{2}, b = 2, c = -\frac{3}{2}$

The equation is: $x^2 + y^2 - \frac{5}{2}x + 2y - \frac{3}{2} = 0$

$$\left[\text{or } \left(x - \frac{5}{4}\right)^2 + (y + 1)^2 = \frac{65}{16}\right]$$

3. Obtain the equation of the circle passing through the three points $(1, 0)$, $(3, 2)$ and $(2, 2+\sqrt{3})$.

[Sol] $x^2+y^2+ax+by+c=0$

Substituting the three points,

substituting $(1, 0)$, $a+c=-1$

substituting $(3, 2)$, $3a+2b+c=-13$

substituting $(2, 2+\sqrt{3})$, $2a+(2+\sqrt{3})b+c=-11-4\sqrt{3}$

Solving for a, b, and c, we can derive the equation.

$\therefore a=-2, b=-4, c=1$

$\therefore$ The equation is: $x^2+y^2-2x-4y+1=0$

$[\text{or } (x-1)^2+(y-2)^2=4]$

4. Obtain the equation of the circle circumscribed about the triangle having the three points $P(3, 5)$, $Q(2, -2)$ and $R(-6, 2)$ as vertices.

[Sol] $x^2+y^2+ax+by+c=0$

Substituting the three points,

substituting $(3, 5)$, $3a+5b+c=-34$

substituting $(2, -2)$, $2a-2b+c=-8$

substituting $(-6, 2)$, $-6a+2b+c=-40$

Solving for a, b, and c, we can derive the equation.

$\therefore a=2, b=-4, c=-20$

$\therefore$ The equation is: $x^2+y^2+2x-4y-20=0$

$[\text{or } (x+1)^2+(y-2)^2=25]$

KUMON

M 186 a

Equations of Circles

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

1. Given circle A, defined as $x^2 + y^2 = 5$ and line B, defined as $3x - y + 1 = 0$:

(1) Find the points of intersection between circle A and line B.

[Sol] $\begin{cases} x^2+y^2-5=0 \ldots ① \\ 3x-y+1=0 \ldots ② \end{cases}$

Solving ① and ② gives the points of intersection,

$$x=-1, \frac{2}{5} \text{ and } y=-2, \frac{11}{5}$$

$\therefore$ The points of intersection are: $(-1,-2), \left(\frac{2}{5}, \frac{11}{5}\right)$

(2) Obtain the equation of a circle passing through the origin and passing through the two points of intersection between circle A and line B.

[Sol] Let the following be the equation of the circle:

$$x^2+y^2+ax+by+c=0 \ldots ③$$

Since ③ passes through the origin: $c=0$ …④

Since ③ passes through $(-1,-2)$: $-a-2b+c=-5$ …⑤

Since ③ passes through $\left(\frac{2}{5}, \frac{11}{5}\right)$: $\frac{2}{5}a+\frac{11}{5}b+c=-5$ …⑥

Solving ④, ⑤ and ⑥,

$$a=15, b=-5, c=0$$

$\therefore$ The equation is: $x^2 + y^2 + 15x - 5y = 0 \left[\text{or } \left(x+\frac{15}{2}\right)^2+\left(y-\frac{5}{2}\right)^2=\frac{125}{2}\right]$

(3) Obtain the equation of a circle passing through $(1, 0)$ and passing through the two points of intersection between circle A and line B.

[Sol] Let the following be the equation of the circle:

$$x^2+y^2+ax+by+c=0 \ldots ③$$

Since ③ passes through $(1, 0)$: $a+c=-1$ …⑦

Since ③ passes through $(-1,-2)$: $-a-2b+c=-5$ …⑧

Since ③ passes through $\left(\frac{2}{5}, \frac{11}{5}\right)$: $\frac{2}{5}a+\frac{11}{5}b+c=-5$ …⑨

Solving ⑦, ⑧ and ⑨,

$$a=3, b=-1, c=-4$$

$\therefore$ The equation is: $x^2 + y^2 + 3x - y - 4 = 0 \left[\text{or } \left(x+\frac{3}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{13}{2}\right]$

2. Fill in each of the boxes below.

(1) $(ax + by + c) + k(dx + ey + f) = 0$ is a line passing through the point of intersection of the following two lines:

$$\begin{cases} ax+by+c=0 \\ \boxed{dx+ey+f}=0 \end{cases}$$

(2) $(x^2 + y^2 - 5) + k(3x - y + 1) = 0$ is a circle passing through the two points of intersection of the following circle and line:

$$\begin{cases} \boxed{x^2+y^2-5}=0 \\ \boxed{3x-y+1}=0 \end{cases}$$

(3) To obtain the equation of a circle passing through the origin and the two points of intersection of $x^2 + y^2 - 5 = 0$ and $3x - y + 1 = 0$, substitute (0, 0) into the equation in (2). This gives:

$(0 + 0 - 5) + k(0 - 0 + 1) = 0$

Thus, $k = \boxed{5}$

$\therefore\ x^2 + y^2 + 15x - 5y = 0$

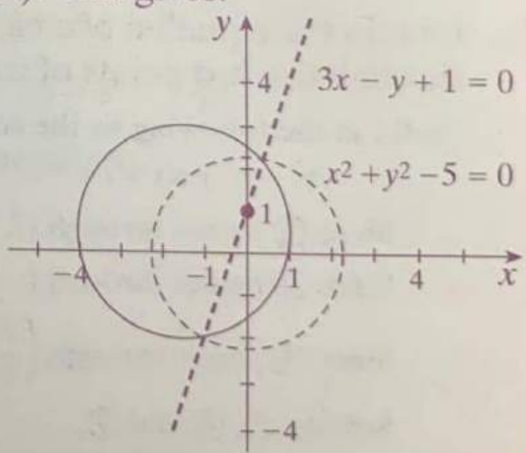
(4) To obtain the equation of a circle passing through (1, 0) and the two points of intersection of $x^2 + y^2 - 5 = 0$ and $3x - y + 1 = 0$, substitute (1, 0) into the equation in (2). This gives:

$k = \boxed{1}$

Therefore,
the equation of the circle is:

$\boxed{x^2 + y^2 + 3x - y - 4 = 0}$

$\left[\text{or } \left(x+\frac{3}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \frac{13}{2}\right]$



Equations of Circles

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

1. Given the equation $(x^2 + y^2 - 4) + k(x^2 + y^2 - 4x - 2y) = 0 \quad \ldots ①$, complete the following exercises.

(1) When $k \neq -1$, ① is a $\boxed{\text{circle}}$ passing through the points of intersection of the following two circles:

$\boxed{x^2 + y^2 - 4} = 0 \qquad \ldots ②$

$\boxed{x^2 + y^2 - 4x - 2y} = 0 \quad \ldots ③$

(2) Obtain the equation of a circle passing through $(-1, 0)$ and the points of intersection of the two circles.

[Sol] Since ① passes through point $(-1, 0)$,

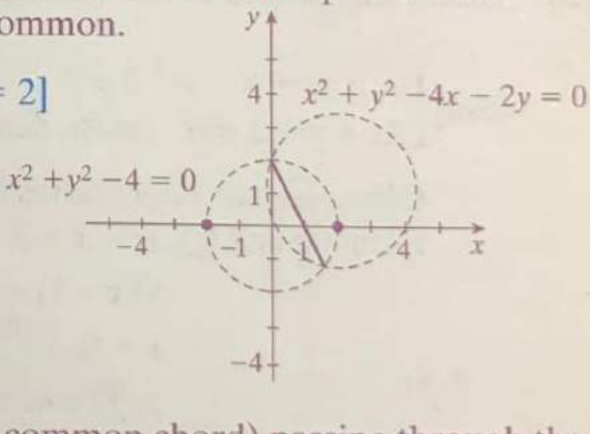
$$k = \frac{3}{5}$$

Substituting this back into ① and rearranging,

$$x^2 + y^2 - \frac{3}{2}x - \frac{3}{4}y = \frac{5}{2} \left[\text{or } \left(x - \frac{3}{4}\right)^2 + \left(y - \frac{3}{8}\right)^2 = \frac{205}{64}\right]$$

(3) When $k = \boxed{-1}$, the original equation represents the line passing through the points of intersection of the two circles. This line is known as the *common chord*, because it connects the points that the two circles share, or have in common.

$\therefore \boxed{2x + y - 2 = 0}$ [or $2x + y = 2$]



2. Obtain the equation of the line (the common chord) passing through the points of intersection of the following two circles:

$$\begin{cases} x^2 + y^2 - 2x - 2y - 8 = 0 \\ x^2 + y^2 - 6x - 4y = 0 \end{cases}$$

[Sol] $x^2 + y^2 - 2x - 2y - 8 - (x^2 + y^2 - 6x - 4y) = 0$

$2x + y = 4$

$\therefore$ The equation is: $2x + y = 4$

3. Given the following two circles:

$$\begin{cases} x^2+y^2+5x+y-6=0 & \dots ① \\ x^2+y^2-x-y-2=0 & \dots ② \end{cases}$$

(1) Obtain the equation of a circle passing through the origin and the points of intersection of the two given circles.

[Sol] Let the following be the equation of the circle in question:

$$x^2+y^2+5x+y-6+k(x^2+y^2-x-y-2)=0 \quad \dots ③$$

$$(k \neq -1)$$

Since this circle passes through point $(0, 0)$,

$$-6-2k=0$$

$$\therefore k=-3$$

Substituting this into ③ and simplifying,

$$x^2+y^2-4x-2y=0 \; [\text{ or } (x-2)^2+(y-1)^2=5]$$

(2) Obtain the equation of the common chord of the two circles.

[Sol] The common chord is given by ③ when $k=-1$.

$$3x+y=2$$

$\therefore$ The equation of the common chord is: $3x+y=2$

(3) Obtain the length of the common chord.

[Sol] $\begin{cases} x^2+y^2-x-y-2=0 & \dots ② \\ 3x+y-2=0 & \dots ④ \end{cases}$

Obtaining the coordinates of the common points,

From ② and ④, $x^2-x=0$

$$x(x-1)=0$$

$$x=0, 1$$

$\therefore$ The common points are: $(0, 2)$ and $(1, -1)$

Using the distance formula,

$$\sqrt{(0-1)^2+(2+1)^2}=\sqrt{10}$$

$\therefore$ The length of the common chord is: $\sqrt{10}$

M 188 a

Equations of Circles

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. Given the following two circles:

$$\begin{cases} x^2+y^2-4=0 & \cdots \text{①} \\ x^2+y^2-8x-6y+k=0 & \cdots \text{②} \end{cases}$$

(1) Find the range of values for k at which ② represents a circle.

[Sol] Rearranging ②,

$$(x-\boxed{4})^2+(y-\boxed{3})^2=\boxed{25-k}$$

$\therefore$ ② represents a circle when $k<25$. This circle has:

center: $(\boxed{4}, \boxed{3})$ and radius: $\boxed{\sqrt{25-k}}$

(2) Investigate the relationship between the two circles.

[Sol]

(case a) When the two circles touch at just one point externally, this happens only when

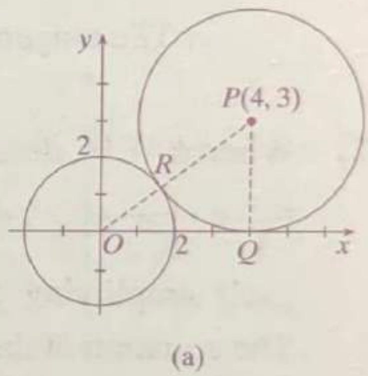
$OQ^2+QP^2=OP^2=(OR+RP)^2$

$4^2+3^2=(\boxed{2}+RP)^2$

Therefore, $RP=3$

Thus, the two circles touch at one point externally when: $\sqrt{25-k}=3$

$\therefore k=16$



(a)

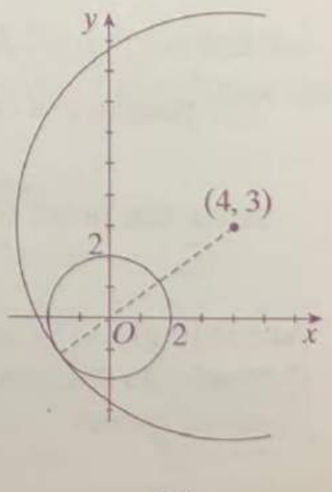
(case b) The two circles do not touch each other when: $\sqrt{25-k}<\boxed{3}$.

Combining conditions,

$\boxed{16}<k<\boxed{25}$

(case c) From the diagram, the two circles touch at just one point internally when: $\sqrt{25-k}=\boxed{7}$

$\therefore k=\boxed{-24}$



(c)

(case d) The two circles intersect at 2 points when: $\boxed{-24}<k<\boxed{16}$

(case e) Circle ① is completely inside circle ② when: $k<\boxed{-24}$

2. Given the same two circles as on side a:

$$\begin{cases} x^2+y^2-4=0 & \dots ① \\ x^2+y^2-8x-6y+k=0 & \dots ② \end{cases}$$

(1) Find the equation of the common chord of the two circles when $k=0$.

[Sol] The equation of the chord is given by eliminating all second-degree terms from ① and ② and then subtracting them:

$$4x+3y=2$$

$\therefore$ The equation of the common chord is: $4x+3y=2$

(2) Obtain the point of tangency of the two circles (where they touch each other) when $k=16$.

[Sol] Solving ① and ② after substituting 16 for k,

$$x=\frac{8}{5},\ y=\frac{6}{5}$$

$\therefore$ The tangent point is: $\left(\frac{8}{5}, \frac{6}{5}\right)$

(3) When $k=16$, describe the following line.

$(x^2+y^2-4)-(x^2+y^2-8x-6y+16)=0 \ \dots ③$

[Sol] Simplifying ③, $4x+3y=10 \ \dots ④$

The equation of the line passing through the centers of the two circles is:

$$y=\boxed{\frac{3}{4}x} \quad \dots ⑤$$

The product of the slopes of ④ and ⑤ is: $-\frac{4}{3}\times\boxed{\frac{3}{4}}=-1$

Also, the point of intersection of ④ and ⑤ is: $(x, y)=\left(\boxed{\frac{8}{5}}, \boxed{\frac{6}{5}}\right)$

Accordingly, ③ is a line which creates a $\boxed{90}^\circ$ angle to the line passing through the centers of the two circles, and which passes through their $\boxed{\text{tangent}}$ point.

Equations of Circles

Time : to : Date Name

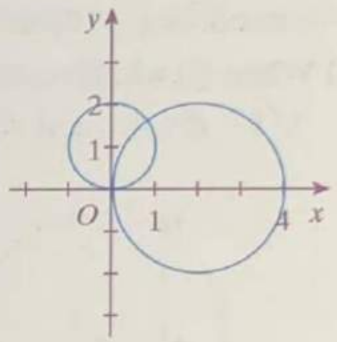
100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

1. For each of the following pairs of circles, graph both and obtain the equation of the line connecting their two points of intersection, where applicable.

(1) $\begin{cases} x^2+y^2=4x & \cdots ① \\ x^2+y^2=2y & \cdots ② \end{cases}$

[Sol] $\begin{cases} (x-2)^2+y^2=4 & \cdots ③ \\ x^2+(y-1)^2=1 & \cdots ④ \end{cases}$

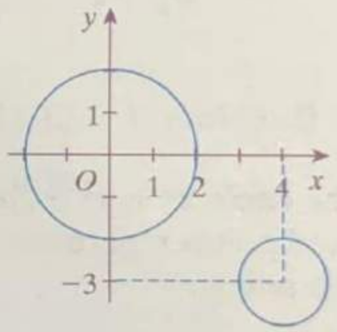
From ① − ②, $y=2x$



(2) $\begin{cases} x^2+y^2=4 & \cdots ① \\ x^2+y^2-8x+6y+24=0 & \cdots ② \end{cases}$

[Sol] From the diagram,
the two circles do not intersect.

∴ There is no line that satisfies the condition.



2. Obtain the equation of the circle passing through the origin and the points of intersection of the circle $x^2+y^2+2x+4y-4=0$ and the line $y-7x=2$.

[Sol] Let the following be the equation of the circle:

$$(x^2+y^2+2x+4y-4)+k(y-7x-2)=0 \quad \cdots ①$$

Since this circle passes through the origin,

$$k=-2$$

Substituting this into ① and simplifying,

$$x^2+y^2+16x+2y=0 \ [\text{or } (x+8)^2+(y+1)^2=65]$$

3. Obtain the range of values of d at which the following two circles do not have any common points. Assume $d > 0$.

$$\begin{cases}(x-2)^2+(y-4)^2=16 & \dots ① \\ (x-1)^2+(y-2)^2=d^2 & \dots ②\end{cases}$$

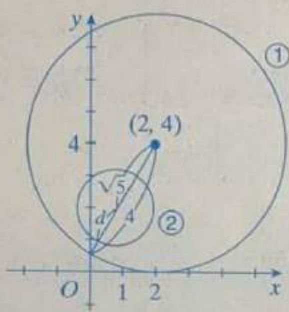
[Sol] The distance between the center of the circles is:

$$\sqrt{(2-1)^2+(4-2)^2}=\sqrt{5}$$

Accordingly, these are the cases where there are no common points:

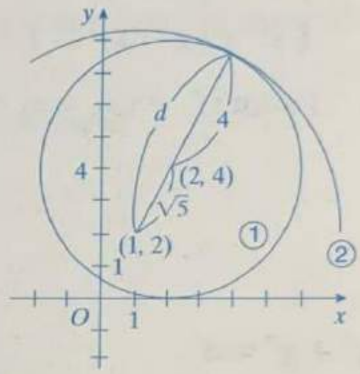
(a) When ① wholly contains ②:

$$\sqrt{5}+d<4 \quad \therefore\ d<4-\sqrt{5}$$



(b) When ② wholly contains ①:

$$d>4+\sqrt{5}$$



Answer: $d<4-\sqrt{5}$ or $d>4+\sqrt{5}$

4. The circle $x^2+y^2-2a(x+y-1)=1$ always passes through two fixed points regardless of the value of a. Find the coordinates of these two points.

[Sol] Rearranging the circle's equation,

$$(x^2+y^2-1)-2a(x+y-1)=0 \quad \dots ①$$

Separating this into two equations,

$$\begin{cases}x^2+y^2-1=0 & \dots ② \\ x+y-1=0 & \dots ③\end{cases}$$

① represents circles passing through the points of intersection of ② and ③. Solving ② and ③ gives the two fixed points:

$$\begin{cases}x=0 \\ y=1\end{cases} \quad \begin{cases}x=1 \\ y=0\end{cases}$$

Therefore, the circle passes through the two fixed points: (0, 1) and (1, 0)

M 190 a Equations of Circles

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

1. Obtain the equation of each of the following:

(1) The circle(s) passing through point $(-8, 1)$ and touching both axes.

[Sol] From the above conditions, the circle must lie in the 2nd Quadrant. Accordingly, its equation is:

$$(x+a)^2+(y-a)^2=a^2$$

Since this passes through point $(-8, 1)$,

$$(-8+a)^2+(1-a)^2=a^2$$

$$(a-5)(a-13)=0$$

$$\therefore\ a=5,\ 13$$

$$\therefore \begin{cases}(x+13)^2+(y-13)^2=169\\(x+5)^2+(y-5)^2=25\end{cases}$$

(2) A circle passing through the three points $(1, 1)$, $(2, -1)$ and $(3, 2)$.

[Sol] Let the following be the equation of the circle:

$$x^2+y^2+ax+by+c=0$$

Since it passes through $(1, 1)$: $a+b+c+2=0$ ... ①

Since it passes through $(2, -1)$: $2a-b+c+5=0$ ... ②

Since it passes through $(3, 2)$: $3a+2b+c+13=0$ ... ③

Solving ①, ② and ③,

$$a=-5,\ b=-1,\ c=4$$

$$\therefore\ x^2+y^2-5x-y+4=0$$

$$\left[\text{or } \left(x-\frac{5}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{5}{2}\right]$$

2. Obtain the equation of the circle(s) passing through point (2, 4) and touching both axes.

[Sol] From the above conditions, the circle must lie in the 1st Quadrant. Accordingly, its equation is:

$$(x-a)^2+(y-a)^2=a^2$$

Since this passes through point (2, 4),

$$(2-a)^2+(4-a)^2=a^2$$

$$(a-2)(a-10)=0$$

$$\therefore a=2,\ 10$$

$$\therefore \begin{cases}(x-2)^2+(y-2)^2=4\\(x-10)^2+(y-10)^2=100\end{cases}$$

3. Prove that the circle $x^2+y^2-2ay=1$ always passes through two fixed points regardless of the value of a.

[Proof] Rearranging the circle's equation,

$$x^2+y^2-1-2ay=0 \quad \ldots ①$$

Separating this into two equations,

$$\begin{cases}x^2+y^2-1=0 & \ldots ②\\ y=0 & \ldots ③\end{cases}$$

Solving ② and ③ gives the two fixed points,

$$\begin{cases}x=1\\y=0\end{cases} \qquad \begin{cases}x=-1\\y=0\end{cases}$$

Since ② and ③ hold for these values, ① holds regardless of the value of a.

Thus, circle ① passes through the two fixed points $(1, 0)$ and $(-1, 0)$.

M 191 a Tangent Lines of Circles

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

In each of the following exercises, obtain the points of intersection of the given circle and line and then draw the graph.

(1) $\begin{cases} x^2+y^2=1 & \dots ① \\ x+y=1 & \dots ② \end{cases}$

[Sol] We can rewrite ② as $y = 1 - x$.

Substituting this new equation into ①,

$x^2 + (1 - x)^2 = 1$

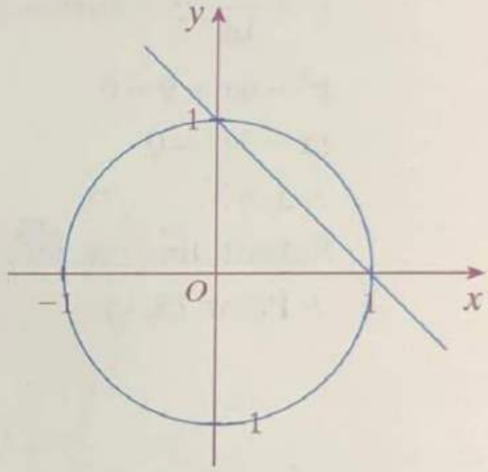
$2x^2 - 2x = 0$

$2x(x - 1) = 0$

$\therefore\ x = 0, 1$

Substituting these points into ②,

$\therefore$ Points: $(0, 1), (1, 0)$



(2) $\begin{cases} x^2+y^2=4 & \dots ① \\ x-2y=2 & \dots ② \end{cases}$

[Sol] We can rewrite ② as $x = 2y + 2$.

Substituting this new equation into ①,

$(2y+2)^2 + y^2 = 4$

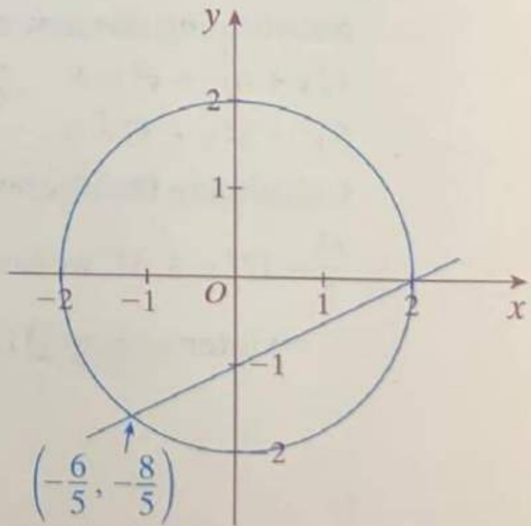
$(5y^2+8y) = 0$

$y(5y+8) = 0$

$\therefore\ y = 0, -\dfrac{8}{5}$

Substituting these points into ②,

$\therefore$ Points: $(2,0), \left(-\dfrac{6}{5}, -\dfrac{8}{5}\right)$



M 191 b

(3) $\begin{cases} x^2+y^2=25 & \ldots ① \\ 3x+4y=25 & \ldots ② \end{cases}$

[Sol] We can rewrite ② as $y=\frac{1}{4}(25-3x)$.

Substituting this new equation into ①,

$x^2+\frac{1}{16}(25-3x)^2=25$

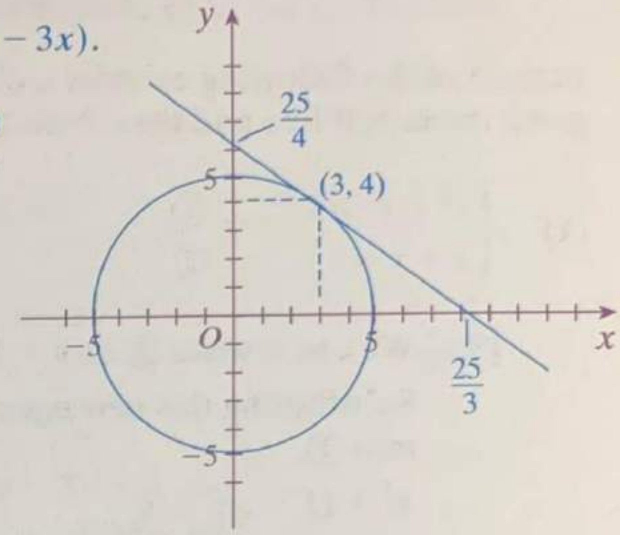
$x^2-6x+9=0$

$(x-3)^2=0$

$\therefore x=3$

Substituting this into ②,

$\therefore$ Point: (3, 4)



(4) $\begin{cases} x^2+y^2=4 & \ldots ① \\ x-2y=6 & \ldots ② \end{cases}$

(Hint) You will have to calculate the discriminant.

[Sol] We can rewrite ② as $x=2y+6$.

Substituting this new equation into ①,

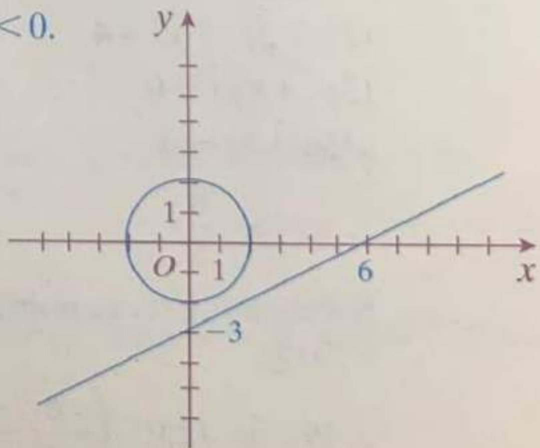
$(2y+6)^2+y^2=4$

$5y^2+24y+32=0$

Calculating the discriminant,

$\frac{D}{4}=12^2-5\cdot 32$, we know that $\frac{D}{4}<0$.

$\therefore$ no intersections [J121]



Tangent Lines of Circles

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

1. Determine the range of values of k at which the line $y = 2x + k$ intersects with the circle $x^2 + y^2 = 9$ at two different points.

(Hint) You will have to calculate the discriminant.

[Sol] $\begin{cases} y = 2x + k & \cdots ① \\ x^2 + y^2 = 9 & \cdots ② \end{cases}$

Substituting ① into ②,

$x^2 + (2x + k)^2 = 9$

$5x^2 + 4kx + (k^2 - 9) = 0$

For the intersection to occur at two points, $\dfrac{D}{4} > 0$

$\therefore\ (2k)^2 - 5(k^2 - 9) > 0$

$4k^2 - 5k^2 + 45 > 0$

$k^2 - 45 < 0$

$(k + 3\sqrt{5})(k - 3\sqrt{5}) < 0$

$\therefore\ -3\sqrt{5} < k < 3\sqrt{5}$

2. Determine the range of values of k at which the line $x + y = -4k$ does not intersect with the circle $x^2 + y^2 = 4$.

[Sol] $\begin{cases} x + y = -4k & \cdots ① \\ x^2 + y^2 = 4 & \cdots ② \end{cases}$

We can rewrite ① as $y = -4k - x$.

Substituting this new equation into ②,

$x^2 + (-4k - x)^2 = 4$

$x^2 + 16k^2 + 8kx + x^2 = 4$

$2x^2 + 8kx + (16k^2 - 4) = 0$

$x^2 + 4kx + (8k^2 - 2) = 0$

For there to be no intersecting points, $\dfrac{D}{4} < 0$

$\therefore\ (2k)^2 - 1 \cdot (8k^2 - 2) < 0$

$4k^2 - 8k^2 + 2 < 0$

Hence, $k^2 > \dfrac{1}{2}$

$\therefore\ k < -\dfrac{\sqrt{2}}{2}$ or $k > \dfrac{\sqrt{2}}{2}$

The positional relationship between a circle and a line can be determined from the discriminant after we eliminate y (or x) from both equations.

When $D > 0$, they intersect at 2 different points.
When $D = 0$, they are tangent and intersect at only 1 point.
When $D < 0$, they do not intersect.

3. Obtain the range of values of m at which the line $y = mx - 3$ intersects with the circle $x^2 + y^2 + 2y = 0$ at two different points.

[Sol] $x^2 + (mx-3)^2 + 2(mx-3) = 0$

$(m^2+1)x^2 - 4mx + 3 = 0$

From the condition of $\frac{D}{4} > 0$, $(-2m)^2 - (m^2+1) \cdot 3 > 0$

$4m^2 - 3m^2 - 3 > 0$

$m^2 > 3$

$\therefore m < -\sqrt{3}, m > \sqrt{3}$

4. Determine the range of values for m at which the line $y = mx$ is tangent to the circle $x^2 + y^2 - 4x + 3 = 0$.

[Sol] $x^2 + (mx)^2 - 4x + 3 = 0$

$(1+m^2)x^2 - 4x + 3 = 0$

From the condition of $\frac{D}{4} = 0$, $(-2)^2 - (1+m^2) \cdot 3 = 0$

$-3m^2 = -1$

$m^2 = \frac{1}{3}$

$\therefore m = \pm\frac{\sqrt{3}}{3}$

Tangent Lines of Circles

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1~

Ex. Obtain the equation of the line passing through the origin and tangent to the circle $x^2 + y^2 - 6x - 2y + 9 = 0$.

[Sol] Because the line has an unknown slope but passes through the origin, we know: $y = mx + (0)$

$\therefore$ Letting $y = mx$ be the equation of the tangent line,

$$\begin{cases} y = mx & \cdots ① \\ x^2 + y^2 - 6x - 2y + 9 = 0 & \cdots ② \end{cases}$$

Substituting ① into ②,

$x^2 + (mx)^2 - 6x - 2(mx) + 9 = 0$

$(1 + m^2)x^2 - 2(3 + m)x + 9 = 0$

For the line and the circle to be tangent, $\frac{D}{4} = 0$

$(-3 - m)^2 - (1 + m^2) \cdot 9 = 0$

$4m^2 - 3m = 0$

$\therefore m = 0, \frac{3}{4}$

$\therefore$ The equations are: $y = 0$ and $y = \frac{3}{4}x$.

1. Obtain the equation of the line passing through point $(0, -3)$ and tangent to the circle $x^2 + y^2 - 2y = 0$.

[Sol] Letting $y = mx - 3$ be the equation of the line,

$$\begin{cases} y = mx - 3 & \cdots ① \\ x^2 + y^2 - 2y = 0 & \cdots ② \end{cases}$$

Substituting ① into ②,

$x^2 + (mx - 3)^2 - 2(mx - 3) = 0$

$(1 + m^2)x^2 - 8mx + 15 = 0$

For the line and the circle to be tangent, $\frac{D}{4} = 0$

$(-4m)^2 - (1 + m^2) \cdot 15 = 0$

$m^2 = 15$

$m = \pm\sqrt{15}$

$\therefore$ The equations are: $y = \pm\sqrt{15}\,x - 3$.

2. Obtain the equation of the line with slope 2 and tangent to the circle $x^2 + y^2 = 9$.

[Sol] Letting $y = 2x + k$ be the equation of the line,

$$\begin{cases} y = 2x + k & \ldots ① \\ x^2 + y^2 = 9 & \ldots ② \end{cases}$$

Substituting ① into ②,

$x^2 + (2x + k)^2 = 9$

$5x^2 + 4kx + (k^2 - 9) = 0$

Since the line and the circle touch at only 1 point, $\frac{D}{4} = 0$.

$\therefore\ (2k)^2 - 5 \cdot (k^2 - 9) = 0$

$4k^2 - 5k^2 + 45 = 0$

$k^2 = 45$

$\therefore\ k = \pm 3\sqrt{5}$

$\therefore$ The equations are: $y = 2x \pm 3\sqrt{5}$.

3. Obtain the equation of the line parallel to $y = -x$ and tangent to the circle $x^2 + y^2 = 18$.

[Sol] Because the line is parallel to $y = -x$, let it be: $y = -x + k$

$$\begin{cases} y = -x + k & \ldots ① \\ x^2 + y^2 = 18 & \ldots ② \end{cases}$$

Substituting ① into ②,

$x^2 + (-x + k)^2 = 18$

$2x^2 - 2kx + (k^2 - 18) = 0$

Since the line and circle touch at only 1 point, $\frac{D}{4} = 0$.

$\therefore\ (-k)^2 - 2 \cdot (k^2 - 18) = 0$

$k^2 - 2k^2 + 36 = 0$

$k^2 = 36$

$\therefore\ k = \pm 6$

$\therefore$ The equations are: $y = -x \pm 6$.

Tangent Lines of Circles

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

1. The equation of the line tangent to the circle $x^2 + y^2 = 10$ at point $P(1, 3)$ on the circle can be obtained using 2 different methods.

(1) Using the discriminant,

The equation of the line passing through $P(1, 3)$ is:

$(y-3) = m(x-1)$

$$\begin{cases} x^2 + y^2 = 10 & \dots ① \\ y = 3 + m(x-1) & \dots ② \end{cases}$$

Substituting ② into ①,

$$x^2 + [3 + m(x-1)]^2 = 10$$
$$(m^2+1)x^2 - 2m(m-3)x + [(m-3)^2 - 10] = 0$$

Since $\dfrac{D}{4} = 0$,

$$[-m(m-3)]^2 - (m^2+1)[(m-3)^2 - 10] = 0$$
$$9m^2 + 6m + 1 = 0$$
$$(3m+1)^2 = 0$$

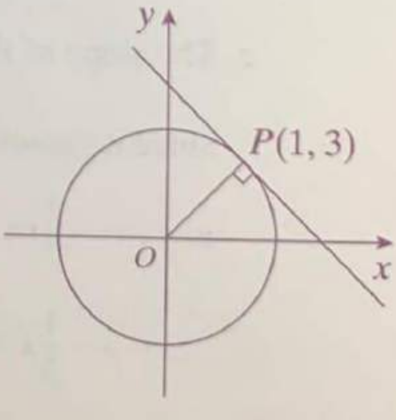
$\therefore m = \boxed{-\dfrac{1}{3}}$

Substituting m into ②, $y = \boxed{-\dfrac{1}{3}}\, x + \boxed{\dfrac{10}{3}}$

(2) Using the fact that the radius and tangent lines are perpendicular,

[Sol] The slope of OP is $\boxed{3}$.

$\therefore$ The slope of the tangent line is: $\boxed{-\dfrac{1}{3}}$.



Since it passes through point (1, 3),

$$y - 3 = -\frac{1}{3}(x-1)$$
$$\therefore y = -\frac{1}{3}x + \frac{10}{3}$$

2. Find the equation of the line tangent to the circle $x^2 + y^2 = 20$ at point $Q(2, 4)$ on the circle using the 2 different methods.

(1) Using the discriminant,

[Sol] The equation of the line passing through $Q(2, 4)$:

$$(y-4) = m(x-2)$$

$$\begin{cases} x^2 + y^2 = 20 & \cdots ① \\ y = 4 + m(x-2) & \cdots ② \end{cases}$$

Substituting ② into ①,

$$x^2 + [4 + m(x-2)]^2 = 20$$

$$(m^2+1)x^2 - 4m(m-2)x + (4m^2 - 16m - 4) = 0$$

Since $\dfrac{D}{4} = 0$,

$$[-2m(m-2)]^2 - (m^2+1)(4m^2-16m-4) = 0$$

$$4m^2 + 4m + 1 = 0$$

$$(2m+1)^2 = 0$$

$$\therefore m = -\frac{1}{2}$$

Substituting m into ②, $y = -\dfrac{1}{2}x + 5$

(2) Using the fact that the radius and tangent lines are perpendicular,

[Sol] The slope of OQ is 2.

$\therefore$ The slope of the tangent line is $-\dfrac{1}{2}$.

Since it passes through point $(2, 4)$,

$$y - 4 = -\frac{1}{2}(x-2)$$

$$\therefore y = -\frac{1}{2}x + 5$$

M 195 a Tangent Lines of Circles

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	–	2~

When $y = mx + n$ is tangent to the circle $x^2 + y^2 = r^2$, then

$$n = \pm r\sqrt{m^2+1}, \qquad r > 0$$

1. In each of the following exercises, obtain the value of k at which the given circle and line are tangent.

(1) $x^2 + y^2 = 9$ and $y = 2x + k$

[Sol] $n = k, r = 3, m = 2$

$k = \pm 3\sqrt{2^2+1}$

$k = \pm 3\sqrt{5}$

(2) $x^2 + y^2 = 9$ and $y = kx + 5$

[Sol] $n = 5, r = 3, m = k$

$5 = \pm 3\sqrt{k^2+1}$

$25 = 9(k^2+1)$

$9k^2 = 16$

$\therefore k = \pm\dfrac{4}{3}$

(3) $x^2 + y^2 = k^2$ and $y = -x - 3$

[Sol] $n = -3, r = k, m = -1$

$-3 = \pm k\sqrt{2}$

$k = \dfrac{\pm 3}{\sqrt{2}}$

$k = \dfrac{3\sqrt{2}}{2}$ $(\because k > 0)$

2. Obtain the value of a at which the line $y - 3 = a(x - 1)$ passing through point $(1, 3)$ on the circumference of the circle $x^2 + y^2 = 10$ becomes tangent to the circle.

[Sol] Rewriting the equation, $y = 3 + a(x - 1) = ax + (3 - a)$

Using the formula of $n = \pm r\sqrt{m^2 + 1}$,

$$3 - a = \pm\sqrt{10}\cdot\sqrt{a^2 + 1}$$

$$(3 - a)^2 = 10(a^2 + 1)$$

$$9 - 6a + a^2 = 10a^2 + 10$$

$$9a^2 + 6a + 1 = 0$$

$$(3a + 1)^2 = 0$$

$\therefore a = -\frac{1}{3}$

3. Obtain the equation of the line drawn from point $(3, 1)$, tangent to the circle $x^2 + y^2 = 5$.

[Sol] The equation of the line passing through $(3, 1)$ is:

$$(y - 1) = a(x - 3)$$

$$y = 1 + ax - 3a$$

Using the formula of $n = \pm r\sqrt{m^2 + 1}$,

$$(1 - 3a) = \pm\sqrt{5}\cdot\sqrt{a^2 + 1}$$

$$(1 - 3a)^2 = 5(a^2 + 1)$$

$$4a^2 - 6a - 4 = 0$$

$$2a^2 - 3a - 2 = 0$$

$$(2a + 1)(a - 2) = 0$$

$\therefore a = -\frac{1}{2}, 2$

$\therefore$ The equations are: $y = 2x - 5$ and $y = -\frac{1}{2}x + \frac{5}{2}$

M 196 a Tangent Lines of Circles

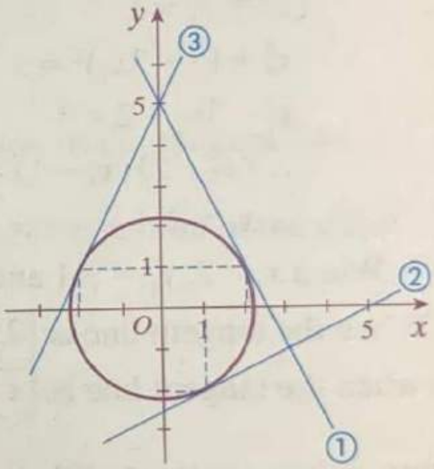
Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	1	2	3~

The equation of the line tangent to the circle $x^2 + y^2 = r^2$ at point $P(x_1, y_1)$ on the circle is:

$$x_1x + y_1y = r^2$$

1. In each of the following exercises, obtain the equation of the tangent line to the circle $x^2 + y^2 = 5$ at the given point, which is on the circumference of the circle, and then draw the graph.



(1) $(2, 1)$

[Sol] $2x + y = 5$ [or $y = -2x + 5$]

(2) $(1, -2)$

[Sol] $x - 2y = 5$ $\left[\text{or } y = \frac{1}{2}x - \frac{5}{2}\right]$

(3) $(-2, 1)$

[Sol] $-2x + y = 5$ [or $y = 2x + 5$]

2. In each of the following exercises, obtain the equation of the line tangent to the circle $x^2 + y^2 = 25$ at the given point.

(1) $P(4, 3)$

[Sol] $4x + 3y = 25$ $\left[\text{or } y = -\frac{4}{3}x + \frac{25}{3}\right]$

(2) $Q(-3, 4)$

[Sol] $-3x + 4y = 25$ $\left[\text{or } y = \frac{3}{4}x + \frac{25}{4}\right]$

(3) $R(-3, -4)$

[Sol] $-3x - 4y = 25$ $\left[\text{or } y = -\frac{3}{4}x - \frac{25}{4}\right]$

3. Obtain the equations of the two lines drawn from point (3, 1) and tangent to the circle $x^2 + y^2 = 5$, and then find the coordinates of the two points of tangency.

[Sol] Letting (x_1, y_1) be the tangent points, the tangent lines are:

$$x_1x + y_1y = \boxed{5}$$

Since both lines pass through the point (3, 1),

$$3x_1 + y_1 = \boxed{5}$$

Also, since (x_1, y_1) is a point on the circumference of the circle,

$$x_1^2 + y_1^2 = \boxed{5}$$

Solving for both equations,

$$\begin{cases} 3x_1 + y_1 = 5 \\ x_1^2 + y_1^2 = 5 \end{cases}$$

$$x_1^2 + (5 - 3x_1)^2 = 5$$

$$x_1^2 - 3x_1 + 2 = 0$$

$$\therefore (x_1 - 2)(x_1 - 1) = 0$$

$$\therefore x_1 = 2, 1$$

When $x_1 = 2$, $y_1 = -1$ and when $x_1 = 1$, $y_1 = 2$

$\therefore$ When the tangent line is: $\boxed{2x - y = 5}$, the tangent point is: $\boxed{(2, -1)}$;

and when the tangent line is: $\boxed{x + 2y = 5}$, the tangent point is: $\boxed{(1, 2)}$.

4. Obtain the equations of the two lines drawn from point $(-1, 3)$ and tangent to the circle $x^2 + y^2 = 2$, and then find the coordinates of the two points of tangency.

[Sol] Since both lines pass through the point $(-1, 3)$,

$$-x_1 + 3y_1 = 2$$

Also, since (x_1, y_1) is a point on the circumference of the circle,

$$x_1^2 + y_1^2 = 2$$

Solving for both equations,

$$(x_1, y_1) = (1, 1) \text{ and } \left(-\frac{7}{5}, \frac{1}{5}\right)$$

$\therefore$ When the tangent line is: $x + y = 2$, the tangent point is: $(1, 1)$;

and when the tangent line is: $-\frac{7}{5}x + \frac{1}{5}y = 2$, the tangent point is: $\left(-\frac{7}{5}, \frac{1}{5}\right)$.

Tangent Lines of Circles

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

1. Obtain the equation of the line tangent to the circle $x^2 + y^2 = 13$ at point $P(2, 3)$ on the circumference of the circle by using each of the following methods:

(1) Using the formula $x_1x + y_1y = r^2$,

[Sol] Since the tangent point is $P(2, 3)$, the equation of the tangent line is:

$$2x+3y=13 \left[\text{or } y=-\frac{2}{3}x+\frac{13}{3}\right]$$

(2) Letting $y - 3 = m(x - 2)$ be the equation of the tangent line,

[Sol] Substituting this new equation into the equation of the circle,

$$x^2+y^2=13$$

$$x^2+[3+m(x-2)]^2=13$$

$$x^2+(3+mx-2m)^2=13$$

$$(m^2+1)x^2+2m(3-2m)x+4(m^2-3m-1)=0$$

Since $\frac{D}{4}=0$,

$$[m(3-2m)]^2-(m^2+1)[4(m^2-3m-1)]=0$$

$$9m^2+12m+4=0$$

$$(3m+2)^2=0$$

$$m=-\frac{2}{3}$$

Substituting this into the equation of the tangent line,

$$y-3=-\frac{2}{3}(x-2)$$

$$y=-\frac{2}{3}x+\frac{13}{3}$$

Ex. Obtain the equation of the line tangent to the circle $(x+3)^2+(y-1)^2=34$ at point $P(2, 4)$ on the circumference of the circle.

[Sol] If we translate (or move) the circle and point P 3 units along the x-axis, and -1 unit along the y-axis, the equation of the circle becomes: $\boxed{x^2+y^2=34}$ and the coordinates of point P become: $\boxed{(5, 3)}$.

Thus, the equation of the tangent line to the new circle and new point P is:

$$\boxed{5x+3y=34}$$

To find the original tangent line, we need to translate it back -3 units along the x-axis, and 1 unit along the y-axis.

$$\therefore\ 5(x+3)+3(y-1)=34$$

$\therefore$ original tangent line is: $\boxed{5x+3y=22}$ $\left[\text{or } y=-\frac{5}{3}x+\frac{22}{3}\right]$

Answers: $x^2+y^2=34$, $(5,3)$, $5x+3y=34$, $5x+3y=22$

2. Obtain the equation of the line tangent to the circle $(x-4)^2+(y+3)^2=10$ at point $Q(1, -4)$ on the circumference of the circle.

[Sol] If we translate (or move) the circle and point Q -4 units along the x-axis, and 3 units along the y-axis, the equation of the circle becomes: $x^2+y^2=10$ and the coordinates of point Q become: $(-3, -1)$.

Thus, the equation of the tangent line to the new circle and new point Q is:

$$-3x-y=10$$

To find the original tangent line, we need to translate it back 4 units along the x-axis, and -3 units along the y-axis.

$$\therefore\ -3(x-4)-(y+3)=10$$

$\therefore$ original tangent line: $3x+y=-1$ [or $y=-3x-1$]

Tangent Lines of Circles

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

1. Obtain the equation of the circle with center at point $(1, 0)$ and tangent to the line $4x - 3y = 1$, and then find the coordinates of the point of tangency.

[Sol] Let $(x-1)^2 + y^2 = r^2$ be the equation of the circle.

$$\begin{cases} 4x - 3y = 1 & \cdots ① \\ (x-1)^2 + y^2 = r^2 & \cdots ② \end{cases}$$

Rewriting ① and then substituting it into ②,

$$(x-1)^2 + \left(\frac{4x-1}{3}\right)^2 = r^2$$

$$25x^2 - 26x + (10 - 9r^2) = 0$$

For the circle and the line to be tangent, $\frac{D}{4} = 0$.

$$13^2 - 25 \cdot (10 - 9r^2) = 0$$

$$r^2 = \frac{9}{25}$$

$\therefore$ The equation of the circle is: $(x-1)^2 + y^2 = \frac{9}{25}$

$\therefore$ The point of tangency between the circle and the line is: $\left(\frac{13}{25}, \frac{9}{25}\right)$

2. Obtain the equation of the circle with center at point $(-4, -1)$ and tangent to the line $x + 2y = 1$, and then find the coordinates of the point of tangency.

[Sol] Let $(x+4)^2 + (y+1)^2 = r^2$ be the equation of the circle.

$$\begin{cases} x + 2y = 1 & \cdots ① \\ (x+4)^2 + (y+1)^2 = r^2 & \cdots ② \end{cases}$$

Rewriting ① and then substituting it into ②,

$$(5 - 2y)^2 + (y+1)^2 = r^2$$

$$5y^2 - 18y + (26 - r^2) = 0$$

For the circle and the line to be tangent, $\frac{D}{4} = 0$.

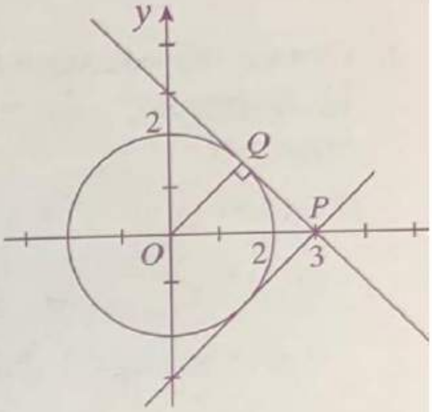
$$9^2 - 5 \cdot (26 - r^2) = 0$$

$$r^2 = \frac{49}{5}$$

$\therefore$ The equation of the circle is: $(x+4)^2 + (y+1)^2 = \frac{49}{5}$

$\therefore$ The point of tangency between the circle and the line is: $\left(-\frac{13}{5}, \frac{9}{5}\right)$

3. Let Q be the point of tangency of the line drawn from point $P(3, 0)$ to the circle $x^2 + y^2 = 4$, and find the length of PQ using the Pythagorean Theorem.



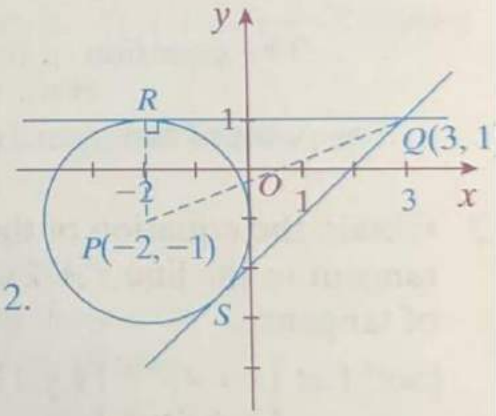
[Sol] Since the radius from the circle's center to Q is perpendicular to the tangent line,
From the graph at right,

$$PO^2 = PQ^2 + QO^2$$

$$3^2 = 2^2 + PQ^2$$

$$\therefore\ PQ = \sqrt{5}$$

4. Obtain the length of the line segment drawn from point $(3, 1)$ to the point of tangency with the circle $x^2 + y^2 + 4x + 2y + 1 = 0$.



[Sol] $x^2 + y^2 + 4x + 2y + 1 = 0$

$$(x^2 + 4x) + (y^2 + 2y) = -1$$

$$(x^2 + 4x + 4) + (y^2 + 2y + 1) = -1 + 4 + 1$$

$$(x + 2)^2 + (y + 1)^2 = 4$$

$\therefore$ The circle has center $P(-2, -1)$ and radius 2.

Let R and S be the tangent points of the line drawn from point $Q(3, 1)$.

Since $RQ = SQ$, we only need to calculate one of the two lengths.

$$PQ^2 = PR^2 + RQ^2$$

$$RQ^2 = PQ^2 - PR^2$$

$$= [(3 + 2)^2 + (1 + 1)^2] - (1 + 1)^2$$

$$= 25$$

$$\therefore\ RQ = 5$$

M 199 a

Tangent Lines of Circles

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

1. Obtain the coordinates of the points of intersection P and Q of the circle $x^2 + y^2 = 1$ and the line $y = -\frac{1}{2}x + 1$, and also the length of PQ.

[Sol] $\begin{cases} x^2 + y^2 = 1 & \cdots ① \\ y = -\frac{1}{2}x + 1 & \cdots ② \end{cases}$

Substituting ② into ①,

$$x^2 + \left(-\frac{1}{2}x + 1\right)^2 = 1$$
$$5x^2 - 4x = 0$$
$$x(5x - 4) = 0$$
$$\therefore\ x = 0, \frac{4}{5}$$

$$\begin{cases} x = 0 \\ y = 1 \end{cases} \quad \begin{cases} x = \frac{4}{5} \\ y = \frac{3}{5} \end{cases}$$

Thus,
$P(0, 1)$ and $Q\left(\frac{4}{5}, \frac{3}{5}\right)$.

$$PQ = \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{3}{5} - 1\right)^2}$$
$$= \frac{\sqrt{16 + 4}}{5}$$
$$= \frac{2\sqrt{5}}{5}$$

2. Obtain the length of the chord created when the line $y = 3x - 10$ passes through the circle $x^2 + y^2 = 20$.

[Sol] $\begin{cases} x^2 + y^2 = 20 & \cdots ① \\ y = 3x - 10 & \cdots ② \end{cases}$

Substituting ② into ①,

$$x^2 + (3x - 10)^2 = 20$$
$$(x - 2)(x - 4) = 0$$
$$\therefore\ x = 2, 4$$

Substituting these into ② gives the points of intersection of ① and ②,

$$\begin{cases} x = 2 \\ y = -4 \end{cases} \quad \begin{cases} x = 4 \\ y = 2 \end{cases}$$

Thus, the length of the chord is:

$$\sqrt{(2-4)^2 + (-4-2)^2} = 2\sqrt{10}$$

3. Given $\begin{cases} x^2+y^2=9 & \ldots ① \\ y=2x+k & \ldots ② \end{cases}$

(1) Determine the range of the value of k at which ① and ② intersect at two different points.

[Sol] Substituting ② into ①,

$$x^2+(2x+k)^2=9$$
$$5x^2+4kx+k^2-9=0$$

For ① and ② to intersect at two different points,

$$\frac{D}{4}=(2k)^2-5\cdot(k^2-9)>0$$
$$k^2<45$$
$$-3\sqrt{5}<k<3\sqrt{5}$$

(2) Given that A and B are the points of intersection of ① and ②, obtain the length of chord AB when $k=-6$.

[Sol] $\begin{cases} x^2+y^2=9 & \ldots ① \\ y=2x-6 & \ldots ② \end{cases}$

Substituting ② into ①,

$$x^2+(2x-6)^2=9$$
$$5x^2-24x+27=0$$
$$(5x-9)(x-3)=0$$
$$\therefore\ x=\frac{9}{5},\ 3$$

Substituting these into ② gives the points of intersection of ① and ②.

$$\begin{cases} x=\dfrac{9}{5} \\ y=-\dfrac{12}{5} \end{cases} \qquad \begin{cases} x=3 \\ y=0 \end{cases}$$

Thus, the length of the chord is:

$$\sqrt{\left(\frac{9}{5}-3\right)^2+\left(-\frac{12}{5}-0\right)^2}=\frac{6\sqrt{5}}{5}$$

M 200 a Tangent Lines of Circles

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

1. Obtain the equation of each of the following circles.

(1) A circle passing through point $(-8, 1)$ and touching both axes.

[Sol] From the above conditions, the circle must lie in the 2nd Quadrant. Thus, its equation is:

$$(x+a)^2+(y-a)^2=a^2$$

Since this passes through point $(-8, 1)$,

$$(-8+a)^2+(1-a)^2=a^2$$

$$(a-5)(a-13)=0$$

$$\therefore a=5, 13$$

$$\therefore \begin{cases} (x+13)^2+(y-13)^2=169 \\ (x+5)^2+(y-5)^2=25 \end{cases}$$

(2) A circle passing through the three points $(1, 1)$, $(2, -1)$, and $(3, 2)$.

[Sol] Let the following be the equation of the circle:

$$x^2+y^2+ax+by+c=0$$

Since it passes through $(1, 1)$,

$$a+b+c+2=0 \quad \dots ①$$

Since it passes through $(2, -1)$,

$$2a-b+c+5=0 \quad \dots ②$$

Since it passes through $(3, 2)$,

$$3a+2b+c+13=0 \quad \dots ③$$

Solving ①, ② and ③,

$$a=-5, b=-1, c=4$$

$\therefore$ The equation of the circle is:

$$x^2+y^2-5x-y+4=0 \left[\text{or } \left(x-\frac{5}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{5}{2}\right]$$

2. Obtain the equation of the line drawn from the origin and tangent to the circle $x^2 + y^2 - 6x - 2y + 8 = 0$.

[Sol] Let $y = mx \ldots$ ① be the equation of the line.

Substituting ① into the equation of the circle,

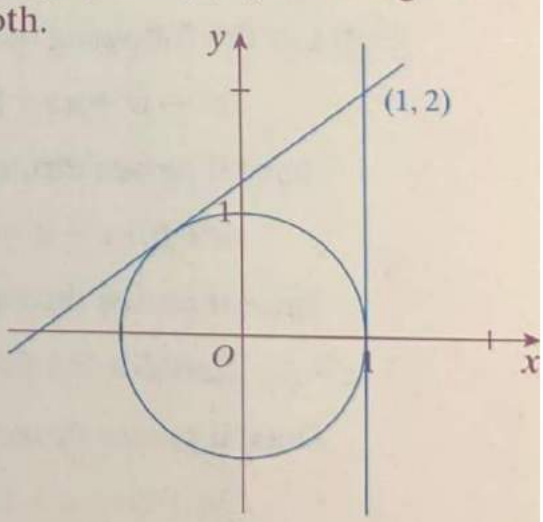
$$x^2 + m^2x^2 - 6x - 2mx + 8 = 0$$
$$(1 + m^2)x^2 - 2(3 + m)x + 8 = 0$$

$$\frac{D}{4} = [(3 + m)]^2 - (1 + m^2) \cdot 8 = 0$$
$$7m^2 - 6m - 1 = 0$$
$$(7m + 1)(m - 1) = 0$$
$$m = -\frac{1}{7}, 1$$

Hence, ① gives the equation of the line:

$$\begin{cases} y = -\frac{1}{7}x \\ y = x \end{cases}$$

3. Obtain the equation of the line passing through point $(1, 2)$ and tangent to the circle $x^2 + y^2 = 1$, and then graph both.



[Sol] Letting (x_1, y_1) be the point of tangency, the equation of the tangent line is:

$$x_1x + y_1y = 1 \quad \ldots ①$$

Since this passes though point $(1, 2)$,

$$x_1 + 2y_1 = 1 \quad \ldots ②$$

Also, since (x_1, y_1) lies on the circle's circumference,

$$x_1^2 + y_1^2 = 1 \quad \ldots ③$$

Solving ② and ③,

$$(x_1, y_1) = (1, 0), \left(-\frac{3}{5}, \frac{4}{5}\right)$$

Substituting these into ①,

$$\begin{cases} x = 1 \\ y = \frac{3}{4}x + \frac{5}{4} \end{cases}$$